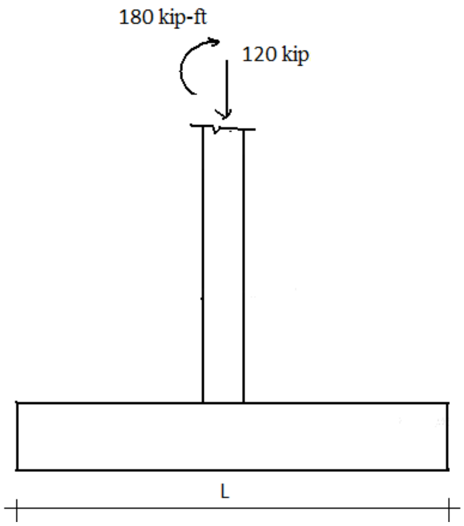
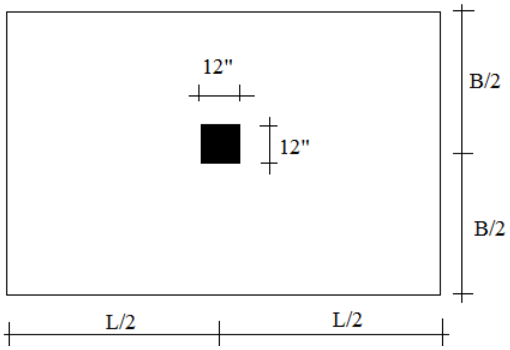
Problem 7.1

Determine the soil pressure distribution under the footing.

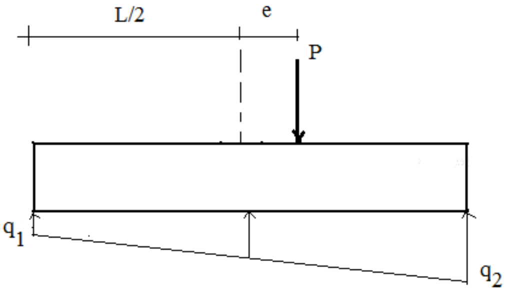


Elevation



Plan

$$e = \frac{M}{P} = \frac{180}{120} = 1.5\ \text{ft}$$



(a) B= L= 8 ft

$$e = 1.5\ \text{ft} < \frac{L}{6} = 1.33\ \text{ft}$$

$$\therefore \quad q_2 = \frac{P}{B\,L} + \frac{6\,(P\,e)}{B\,L^2} = \frac{180}{8(8)} + \frac{6\,(180)}{8^3} = 4.92\ \text{kip/ft}^2$$

$$q_1 = \frac{P}{B\,L} - \frac{6\,P\,e}{B\,L^2} = \frac{180}{8(8)} - \frac{6\,(180)}{8^3} = 0.7\ \text{kip/ft}^2$$

(b) L=10 ft B =5 ft

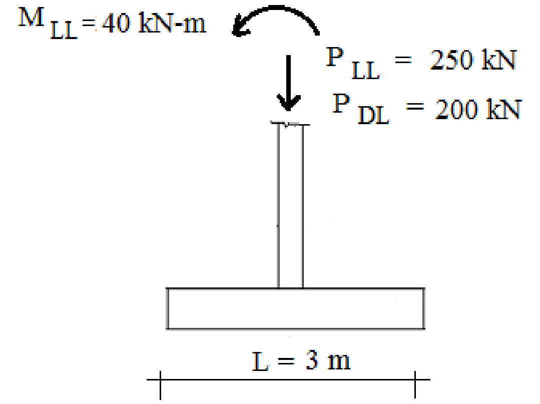
$$e = 1.5\ \text{ft} < \frac{L}{6} = 1.66\ \text{ft}$$

$$\therefore \quad q_2 = \frac{P}{B\,L} + \frac{6\,(P\,e)}{B\,L^2} = \frac{180}{5(10)} + \frac{6\,(180)}{(5)(10)^2} = 5.76\ \text{kip/ft}^2$$

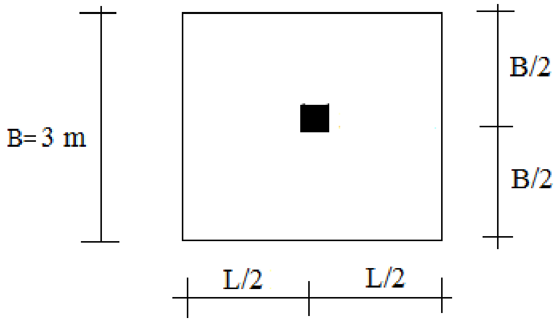
$$q_1 = \frac{P}{B\,L} - \frac{6\,P\,e}{B\,L^2} = \frac{180}{5(10)} - \frac{6\,(180)}{(5)(10)^2} = 1.42\ \text{kip/ft}^2$$

Problem 7.2

Determine the soil pressure distribution under the footing. Use a factor of 1.2 for DL and 1.6 for LL.



Elevation



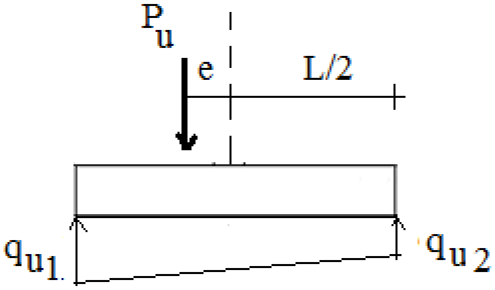
Plan

$$P_u = 1.2\,P_D + 1.6\,P_L = 1.2(200) + 1.6(250) = 640\ \text{kN}$$

$$M_u = 1.2\,M_D + 1.6\,M_L = 0 + 1.6(64) = 64\ \text{kN-m}$$

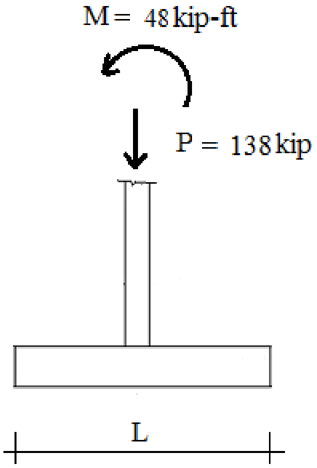
$$e = \frac{M_u}{P_u} = \frac{64}{640} = 0.1 \text{ m} < \frac{L}{6} = 0.5 \text{ m}$$

$$\therefore \begin{aligned} q_{u1} &= \frac{P_u}{B\,L} + \frac{6\,(P_u\,e)}{B\,L^2} = \frac{640}{3(3)} + \frac{6\,(64)}{3^3} = 85.32 \text{ kN/m}^2 \\ q_{u2} &= \frac{P_u}{B\,L} - \frac{6\,P_u\,e}{B\,L^2} = \frac{640}{3(3)} - \frac{6\,(64)}{3^3} = 56.88 \text{ kN/m}^2 \end{aligned}$$

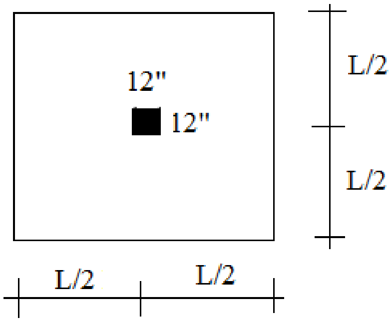


Problem 7.3

The effective soil pressure is 4 kip/ft^2. Determine the maximum allowable value for L.



Elevation



Plan

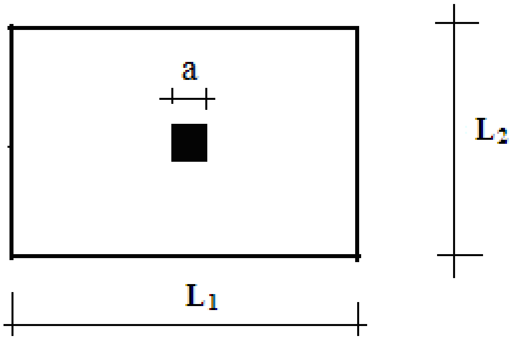
$$e = \frac{M}{P} = \frac{48}{138} = .3478 \text{ ft}$$

$$\text{Assume } e < \frac{L}{6} \qquad \frac{P}{L^{\,2}} + \frac{6\,M}{L^3} = \frac{138}{L^2} + \frac{6\,(48)}{L^3} \le 4$$

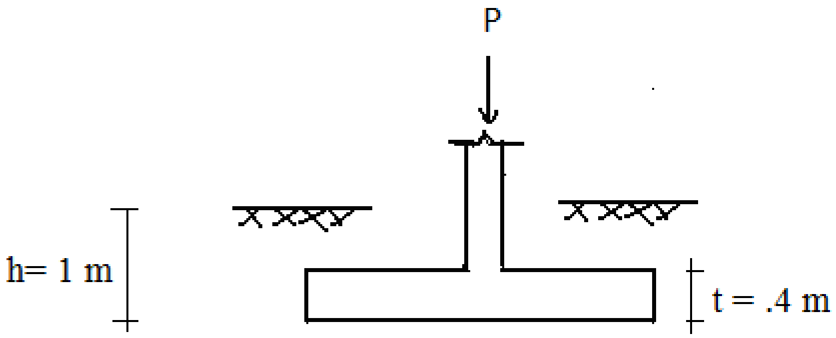
$$4L_{req}{}^3 - 138L_{req} - 288 = 0 \qquad L \ge 6.72 \text{ ft}$$

Problem 7.4

The allowable soil pressure is $q_{allowable} = 250\ \text{kN/m}^2$, $\gamma_{soil} = 18\ \text{kN/m}^3$, $\gamma_{conc} = 24\ \text{kN/m}^3$, P_D = 1000 kN and P_L = 1400 kN.



Plan



Elevation

$$P_{D+L} = 1000 + 1400 = 2400\ \text{kN}$$

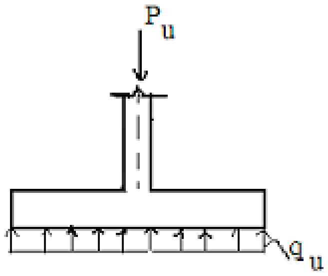
$$q_{effective} = q_{allowable} - \gamma_{conc} t - \gamma_{soil} (h\text{-}t) = 250 - 24(.4) - 18(1 - .4) = 229.6\ \text{kN/m}^2$$

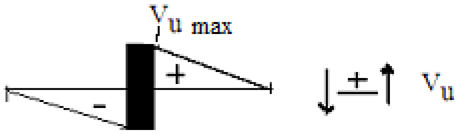
$$P_u = 1.2\ P_D + 1.6\ P_L = 1.2(1000) + 1.6(1400) = 3440\ \text{kN}$$

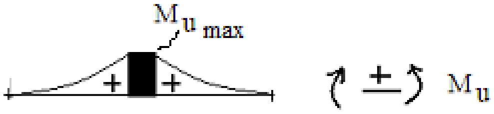
$$q_u = \frac{P_u}{L_1\ L_2}$$

$$V_u(x) = L_2\ q_u\ (\frac{L_1}{2} - \frac{a}{2})$$

$$M_u(x) = \frac{L_2\ q_u}{2}\left(\frac{L_1}{2} - \frac{a}{2}\right)^2$$







(a) A square footing ($L_1 = L_2 = L$)

$$\frac{P_{D+L}}{L^2} \le q_{effective} = 229.6 \text{ kN/m}^2$$

$$L \ge \sqrt{\frac{2400}{229.6}} = 3.23 \text{ m} \qquad \text{use } L = 3.25 \text{ m}$$

$$q_u = \frac{P_u}{L^2} = \frac{3440}{3.25(3.25)} = 326 \text{ kN/m}^2$$

$$V_{u\ max} = L\ q_u\ (\frac{L}{2} - \frac{a}{2}) = (3.25)(326)(1.625 - .225) = 1483 \text{ kN}$$

$$M_{u_{max}} = \frac{L\ q_u}{2}\left(\frac{L}{2} - \frac{a}{2}\right)^2 = \frac{(3.25)(326)}{2}(1.625 - .225)^2 = 1038 \text{ kN-m}$$

(b)A rectangular footing with $L_2 = 2.5$ m.

$$\frac{P_{D+L}}{L_1(L_2)} \le q_{effective} = 229.6 \text{ kN/m}^2$$

$$L_1 \ge \frac{2400}{2.5(229.6)} = 4.18 \text{ m} \qquad \text{use } L_1 = 4.2 \text{ m}$$

$$q_u = \frac{P_u}{L_1\ L_2} = \frac{3440}{2.5(4.2)} = 327.6\ \text{kN/m}^2$$

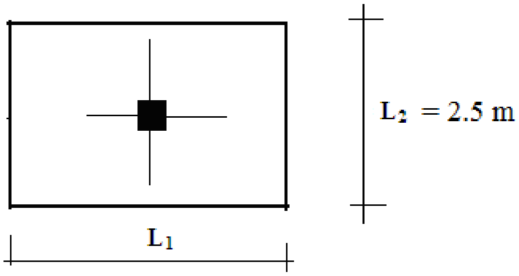
$$V_{u\ max} = L_2\ q_u\ (\frac{L_1}{2} - \frac{a}{2}) = (2.5)(327.6)(2.1 - .225) = 1536\ \text{kN}$$

$$M_{u_{max}} = \frac{L_2\ q_u}{2}\left(\frac{L_1}{2} - \frac{a}{2}\right)^2 = \frac{(2.5)(327.6)}{2}(2.1 - .225)^2 = 1440\ \text{kN-m}$$

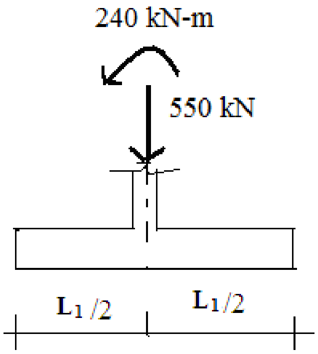
Problem 7.5

A 350 mm x 350 mm column is to be supported on a shallow foundation. Determine the dimensions (either square or rectangular) for the following conditions. The effective soil pressure is $q_{effective} = 180\ \text{kN/m}^2$.

(a) The centerline of the column coincides with the centerline of the footing.



Plan



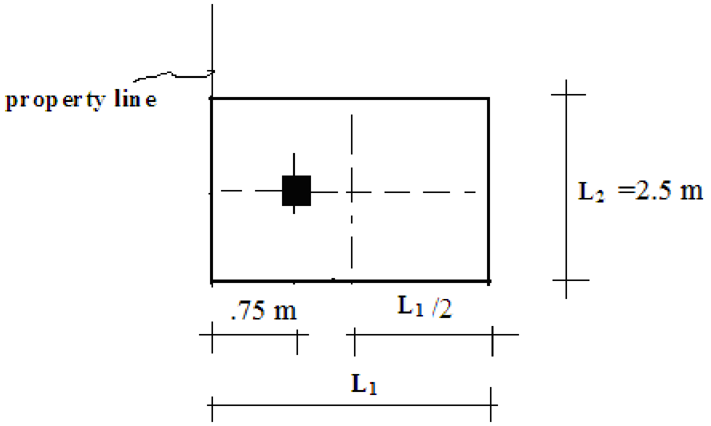
Elevation

$$q = \frac{P}{L_1\ L_2} + \frac{6\ (P\ e)}{L_2\ L_1^{\ 2}} \leq q_{effective}$$

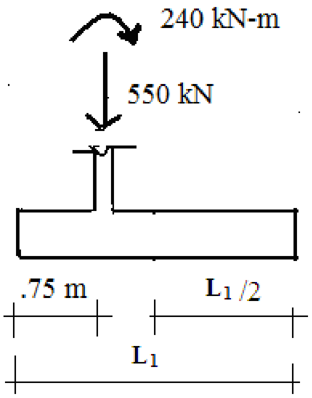
$$\frac{240}{2.5L_1} + \frac{6\ (550)}{2.5L_1^{\ 2}} \leq 180$$

$$180L_1^{\ 2} - 220L_1 - 576 = 0 \qquad \rightarrow \qquad L_1 = 2.5\ \text{m}$$

(b) The center line of the column is 0.75meter from the property line.



Plan

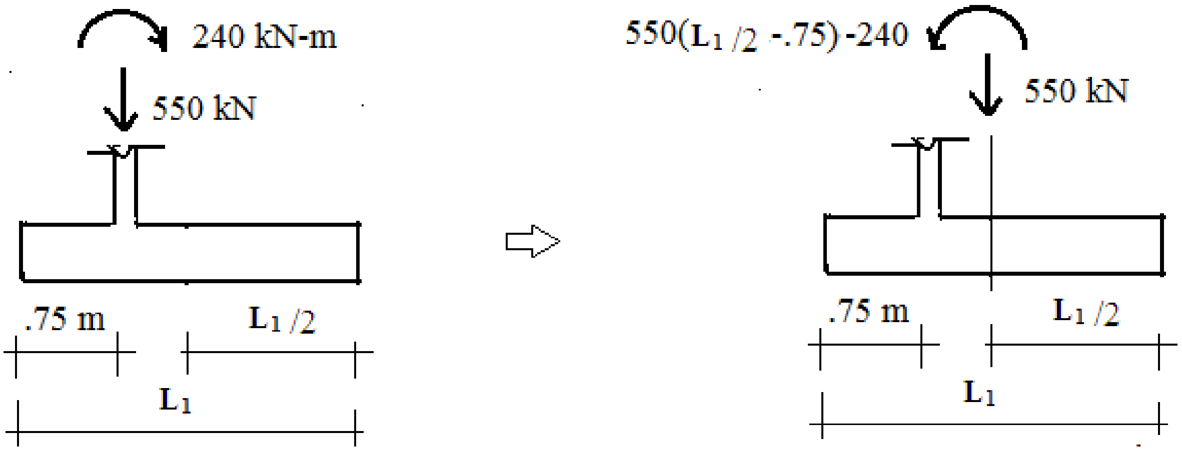


Elevation

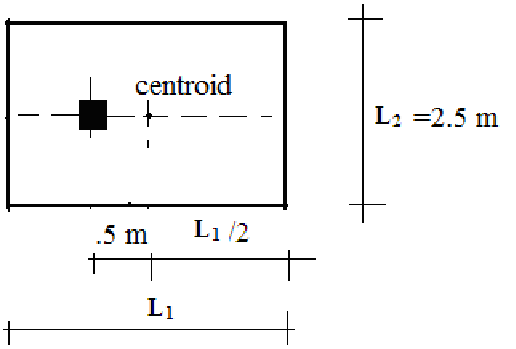
$$q \;=\; \frac{P}{L_1\ L_2} + \frac{6\ (P\ e)}{L_2\ L_1^{\,2}} \le q_{\text{effective}}$$

$$\frac{550}{2.5L_1} + \frac{6\left((\frac{L_1}{2} - .75)(550) - 240\right)}{2.5L_1^{\,2}} \le 180$$

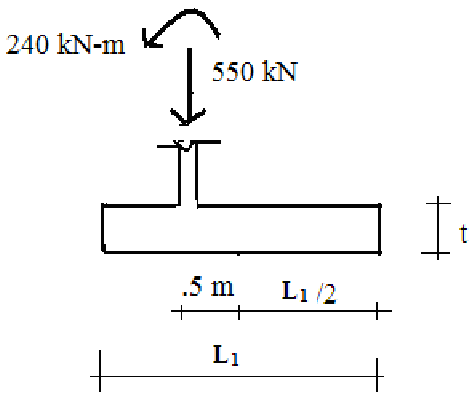
$$180\ L_1^{\,2} - 880\ L_1 + 652.5 = 0 \qquad \rightarrow \qquad L_1 = 3.97\ \text{m} \qquad \text{use } L_1 = 4\ \text{m}$$



(c) The center line of the column is 0.5 meter from the centroid of the footing.



Plan

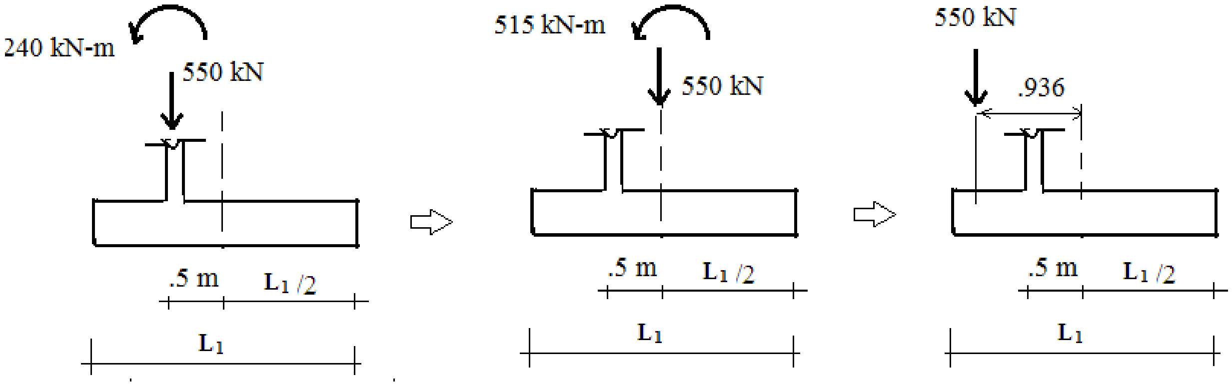


Elevation

$$q = \frac{P}{L_1\ L_2} + \frac{6\ (P\ e)}{L_2\ L_1^2} \le q_{\text{effective}}$$

$$\frac{550}{2.5L_1} + \frac{6\ (515)}{2.5L_1^2} \le 180$$

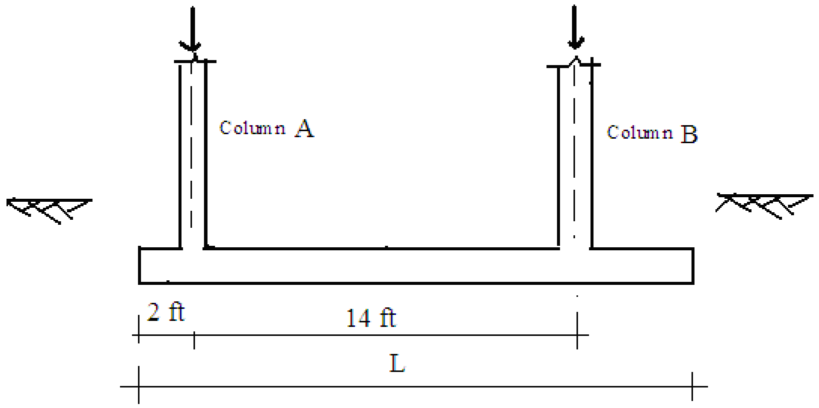
$$180\ L_1^2 - 220\ L_1 - 1236 = 0 \qquad \rightarrow \qquad L_1 = 3.3\ \text{m}$$



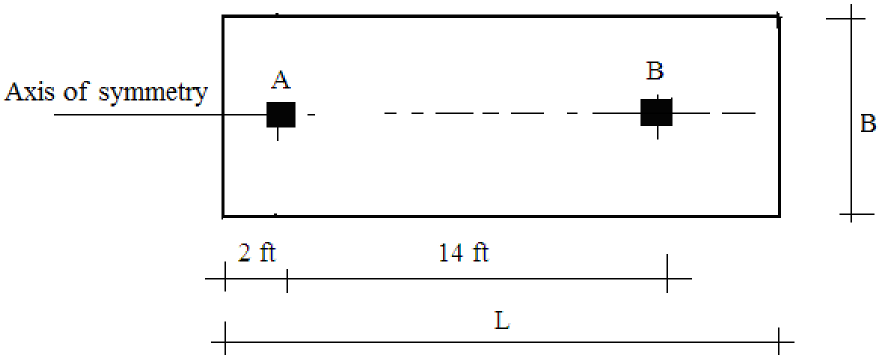
Problem 7.6

A combined footing supports two square columns: Column A is 14 inches x 14 inches and carries a dead load of 140 kips and a live load of 220 kips. Column B is 16 inches x 16 inches and carries a dead load of 260 kips and a live load of 300 kips. The effective soil pressure is $q_e = 4.5\ \text{k/ft}^2$. Assume the soil pressure distribution is uniform. Determine the footing dimensions for the following geometric configurations. Establish the shear and moment diagrams corresponding to the factored loading, $P_u = 1.2\ P_D + 1.6\ P_L$.

Case (**a**):



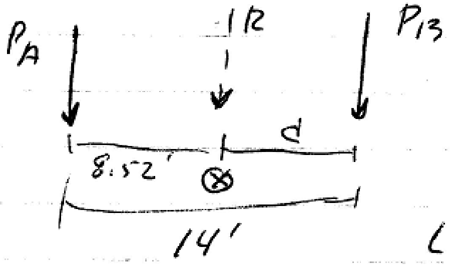
Elevation



Plan

$P_A = 140 + 220 = 360^k$

$P_B = 260 + 300 = 560^k$



$q_{eff} = 4.5\ ^{k/'^2}$

$R = 360 + 560 = \cancel{860}^k$ 920^k

$14\ P_A = R d$

$d = \frac{14(360)}{920} = 5.48'$

Locate centroid at R.

$$A = \frac{926}{4.5} = 204.4\ ft^2$$

$$\frac{L}{2} = 8.52 + 2.0 = 10.52'$$

$$\left.\begin{aligned} L &= 21.04\ ft \\ B &= \frac{204.4}{21.04} = 9.72' \end{aligned}\right\} *$$

Ultimate Load

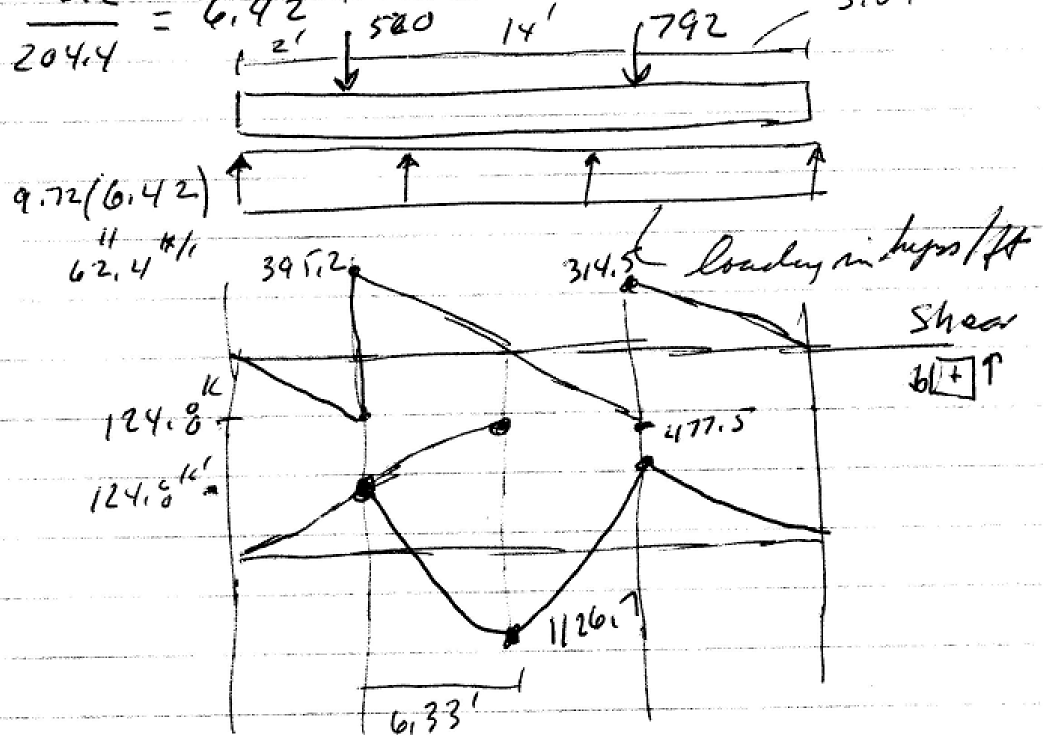
$$P_A = 1.2(140) + 1.6(220) = 520$$

$$P_B = 1.2(260) + 1.6(300) = 792$$

$$R = 1312^k \qquad d = \frac{14(520)}{1312} = 5.55'$$

Slight eccentricity – assume uniform pressure

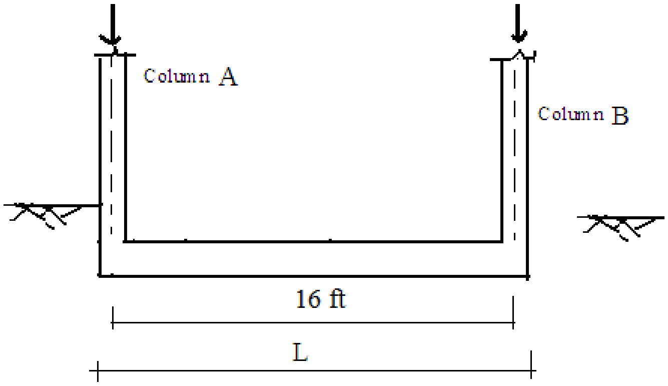
$$q = \frac{1312}{204.4} = 6.42^{k/'^2}$$



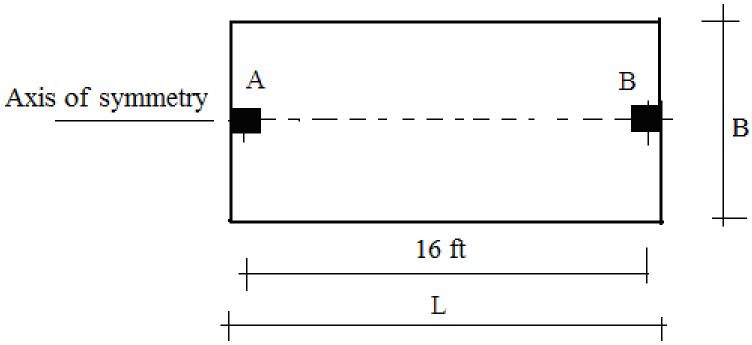
$$M_{max} = \underset{2164.9}{\frac{62.4(8.33)^2}{2}} - \underset{3291.6}{520(6.33)}$$

$$M_{max} = -1126.7^{k'}$$

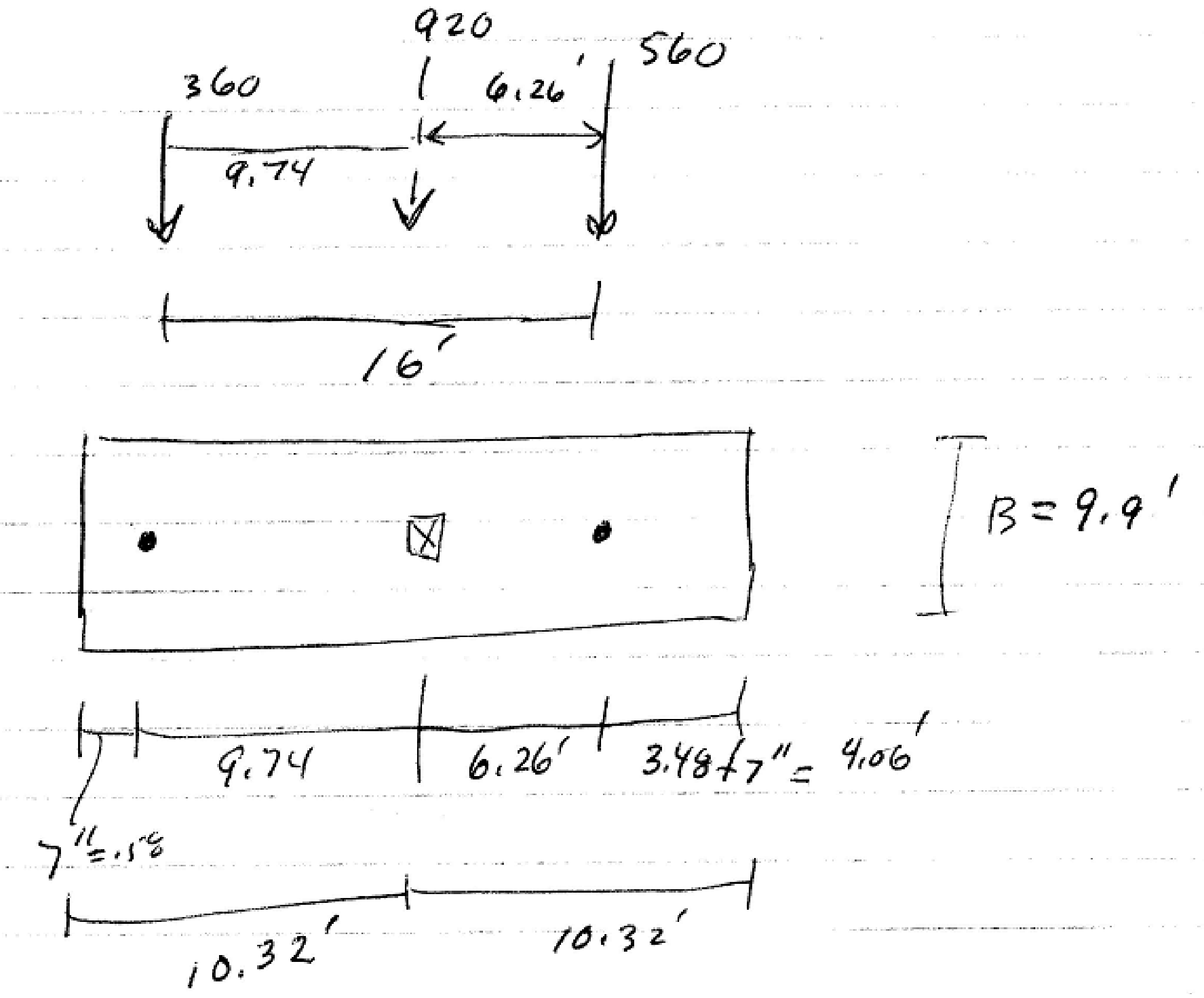
Case (**b**):



Elevation



Plan



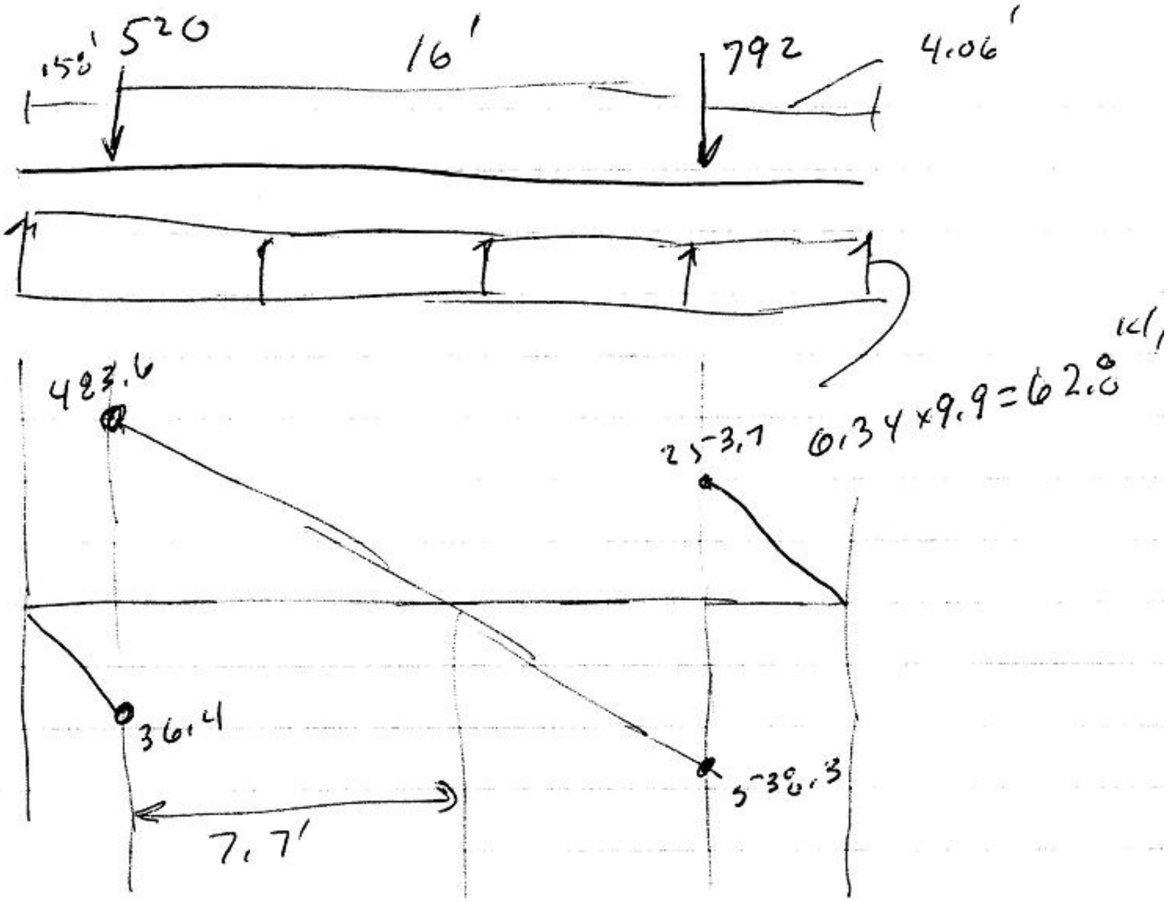
$$A = \frac{920}{4.5} = 204.4\ ft^2$$

$$B = \frac{204.4}{20.64} = 9.9'$$

Ultimate load

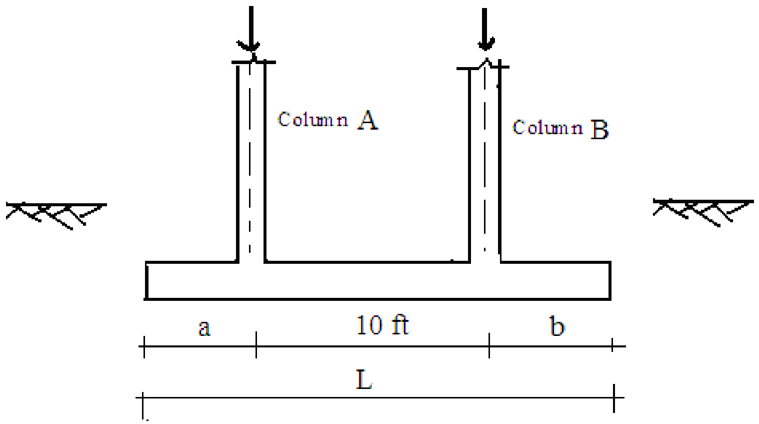
$$d = \frac{16(520)}{\cancel{920}\ 1312} = 6.34'$$

Small change – assume uniform pressure = $\frac{1312}{204.4} = 6.42^{k/'^2}$

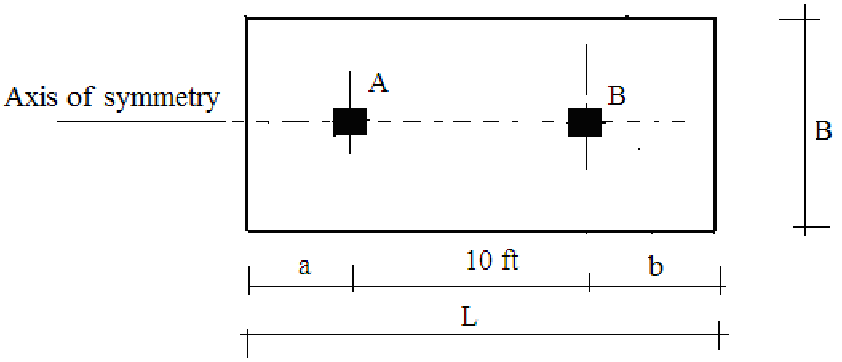


$$M_{max} = 520(7.7) - 62.0\frac{(8.20)^2}{2} = 1851.3$$

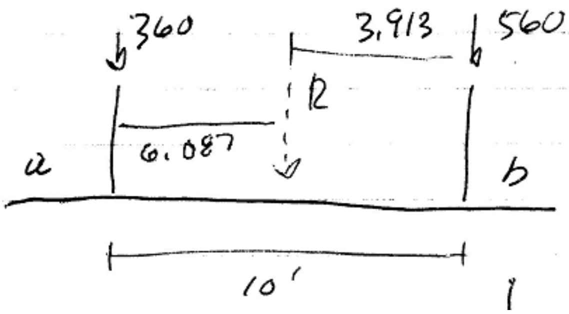
Case (**c**):



Elevation



Plan



$R = 920$

$R \cdot d = 10\ (360)$

$d = 3.913$

Note: Figure incorrect – one a should be a b for uniform pressure.

$A = (10+a+b)(B) = 204.4$

Also, $6.087 + a = 3.913 + b$ for centroid to be on the line of action of R.

$\therefore \; b = a + 2.174$

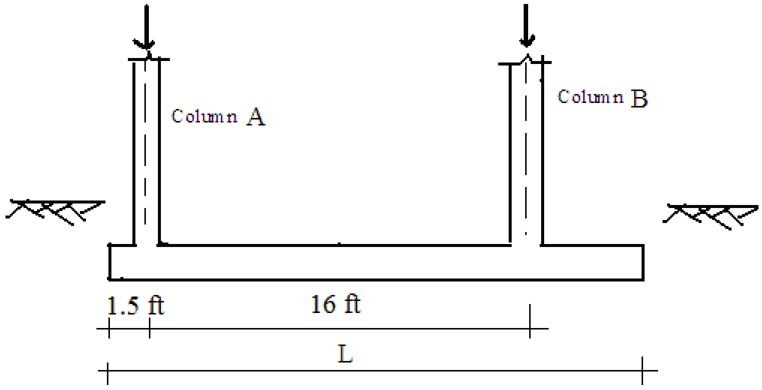
Then $(10 + 2.174 + 2a) B = 204.4$

Choose a. Then determine b.

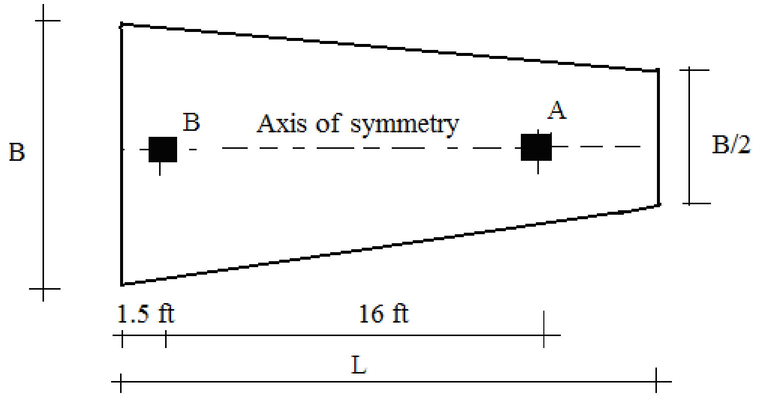
$a = 3'$ $\qquad B = \dfrac{204.4}{18.174} = 10.91'$

$b = 5.174'$

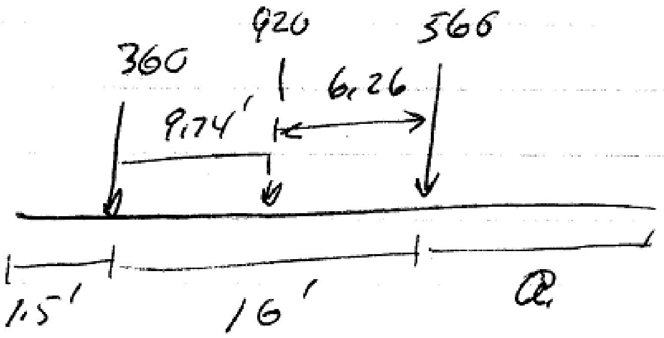
Case (**d**):



Elevation



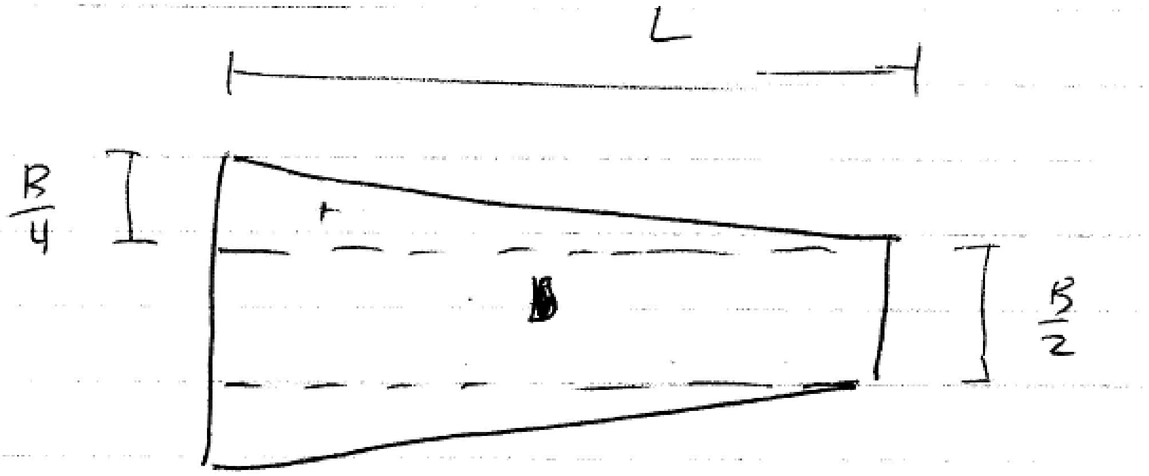
Plan



$$\frac{360(16)}{920} = 6.26'$$

$$6.26 + a = 9.74 + 1.5$$ } for rectangular shape

$$a = 4.98'$$



Centroid

$9.74 + 1.5 = 11.24'$

$\frac{B}{2}L \cdot \frac{L}{2} + 2\left(\frac{L}{3}\right)\left[\frac{1}{2}L \cdot \frac{B}{4}\right] = \left[\frac{B}{2}L + 2\left(\frac{1}{2}L\frac{B}{4}\right)\right] 11.24$

$= \left(\frac{3}{4}BL\right) 11.24$

$BL^2\left[\frac{1}{4} + \frac{1}{12}\right]$

$\frac{1}{3}BL^2 = BL\left(\frac{3}{4}\right)11.24$

$L = \frac{9}{4}(11.24) = 25.29$ ans

Area

$A = 204.4 = \frac{3}{4}BL$

$(25.29)B = 272.5$

$B = 10.8'$ ans

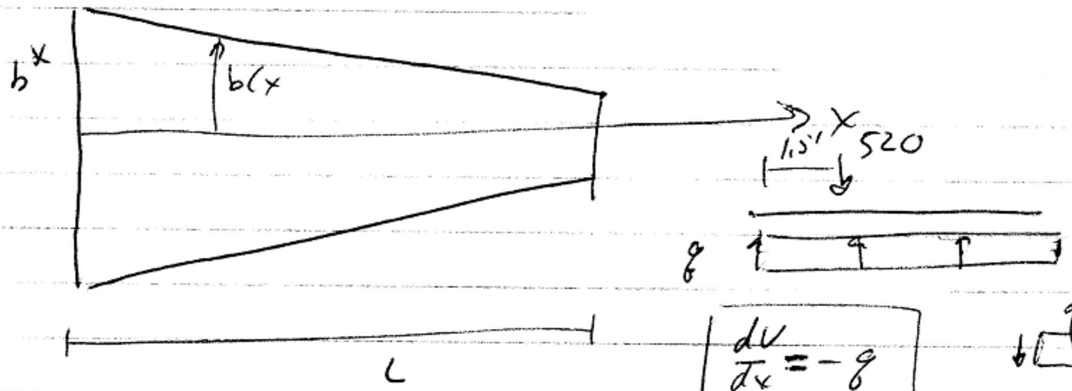
Shear & Moment Distribution - Factored Loads

Assume uniform pressure

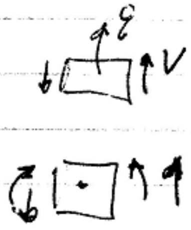
$q' = 6.42 \ ^{k/\prime 2}$

$b^x = 5.4'$ $L = 25.29'$

$b(x) = b^x\left(1 - \frac{x}{L}\right)$



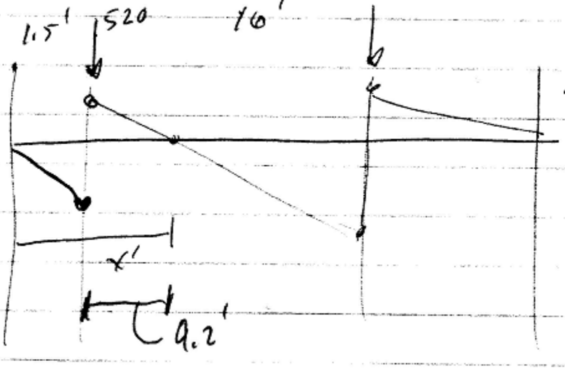
$$\boxed{\begin{array}{l} \frac{dV}{dx} = -q \\ \frac{dM}{dx} = -V \end{array}}$$



$\overset{\uparrow}{q} = 6.42\left[b^x\left(1 - \frac{x}{L}\right)\right] \times 2$

Due to pressure

$$\left\{ \begin{array}{l} V(x) = 6.42\left[b^x\left(-x + \frac{x^2}{2L}\right)\right] \times 2 \\ M(x) = 6.42\left[b^x\left(+\frac{x^2}{2} * \frac{x^3}{6L}\right)\right] \times 2 \end{array} \right.$$



Peak moment $\boxed{V_{net} = 0}$

$2 \times 6.42\left[b^x\left(-x + \frac{x^2}{2L}\right)\right] = -520$

$-34.67\left[x - \frac{x^2}{50.58}\right] = \frac{-520}{2}$

$x - \frac{x^2}{50.58} = \frac{15}{2} = 7.5$

$$x^2 - 50.58x + \overset{379.3}{\cancel{758.6}} = 0$$

$$x = \frac{+50.58 \pm [2558.3 + \overset{1517}{3034}]^{1/2}}{2} = \frac{50.58 \pm 32.2}{2}$$

$$x' = 41.43;\ 9.2'$$

Plot $\left.\begin{array}{l} M(x) - (x - 1.5)520 \\ V(x) + 520 \end{array}\right\}$ for

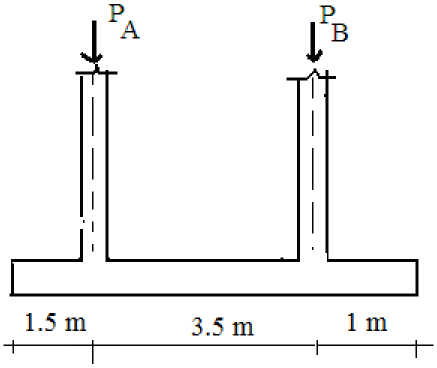
$x = 1.5 \longrightarrow 17.5$

Note: Not the typical distribution for shear and moment due to variable width

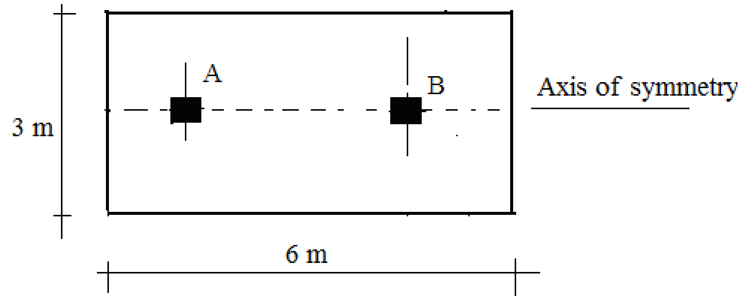
M_{max} @ 9.2' from <u>end</u>

Problem 7.7

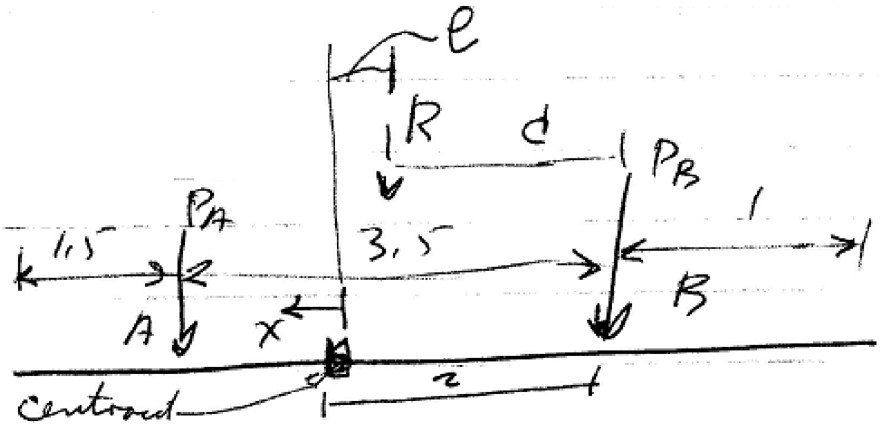
Column A is 350 mm x 350 mm and carries a dead load of 1300 kN and a live load of 450 kN. Column B is 450 mm x 450 mm and carries a dead load of 1400 kN and a live load of 800 kN. The combined footing shown below is used to support these columns. Determine the soil pressure distribution and the shear and bending moment distributions along the longitudinal direction corresponding to the factored loading, $P_u = 1.2\ P_D + 1.6\ P_L$.



Elevation



Plan



$$P_A = 1300 + 450 = 1750$$

$$P_B = 1400 + 800 = 2200$$

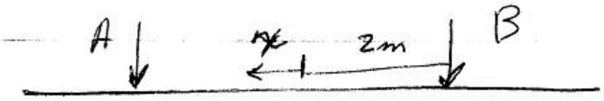
$$R = 1750 + 2200 = 3950$$

$$Rd = 3.5\,(1750) \qquad d = 1.551\,m$$

$$e = 2.0 - 1.551 \quad = -.45\,m \text{ (negative)}$$

$$A = 18\,m^2 \qquad I = \frac{3}{12}(6)^3 = 1.5(36) = 54\,m^4$$

$$q = \frac{R}{A} + \frac{R \cdot e}{I} x$$ linear varying distribution



$$q = \frac{R}{18} + \frac{(e) R}{54} x$$

Ultimate Load

$$P_A = 1.2(1300) + 1.6(450) = 2280$$

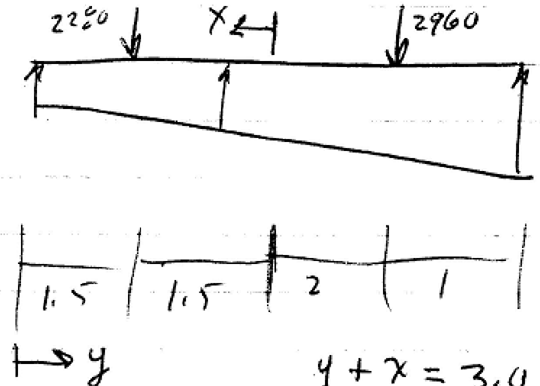
$$P_B = 1.2(1400) + 1.6(800) = 2960$$

$$R = 5240$$

$$d = \frac{2280(3.5)}{5240} = 1.523$$

$e = 2 - 1.523 = -.477$ (negative

$q(x) = \frac{5240}{18} + \frac{5240(-.477)}{54}x = 291.1 - 46.3x$ ~~4H~~ kN/m^2



$y + x = 3.0 \qquad y = 3 - x$

$q(y) = [291.1 - 46.3(3 - y)]3 \qquad kN/m$

$q(y) = [152.2 + 46.3y]3$

$\frac{dV}{dx} = -q \qquad \frac{dM}{dx} = -V$

$\frac{1}{3}V(y) = -152.2y \quad 23.15y^2$

$\frac{1}{3}M(y) = 76.1y^2 + \frac{23.15}{3}y^3$

For pressure loading only. Need to add term due to concentrated loads.

Max Moment $\quad 1.5 < y <$ ~~3.5~~ 5.0

$V(y) + 2280 = 0 \quad$ at M_{max}

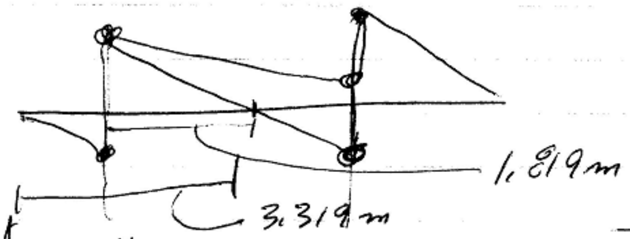
$$3\left[23.15y^2+152.2y\right]-2280=0$$

$$y^2+6.575y-\frac{98.5}{3}=0$$

$$y_m=\frac{1}{2}\left\{-6.575\pm\left[43.23+\frac{394}{3}\right]^{1/2}\right\}$$

$$y_m=\frac{1}{2}\left\{-6.575\overset{+13.212}{\pm\cancel{20.91}}\right\}=\frac{6.637}{2}\cong 3.319\text{ m }\cancel{\text{ft}}$$

Distribution is typical for shear



$$M_{max}=M(y)-2280(1.819)$$

$$=3\left\{838.3+\frac{246.4}{3}\right\}$$

$$\cancel{5654.1}\quad -2280(1.819)$$

$$=\cancel{15462.4}-4147.3$$

$$3361.3$$

$$=\cancel{+986.8}$$

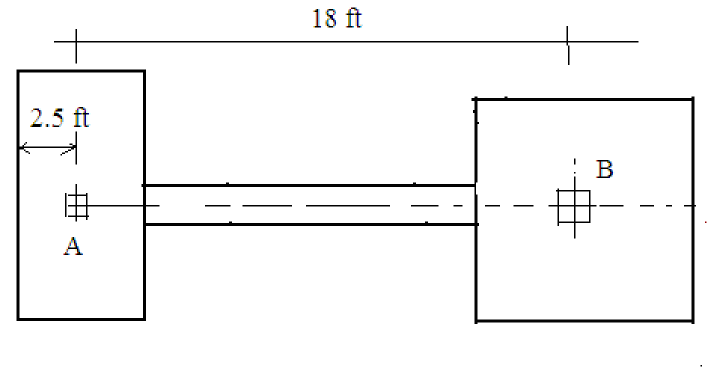
$$M_{max}=-786.0\text{ kN·m ans}$$

Left out factor for width

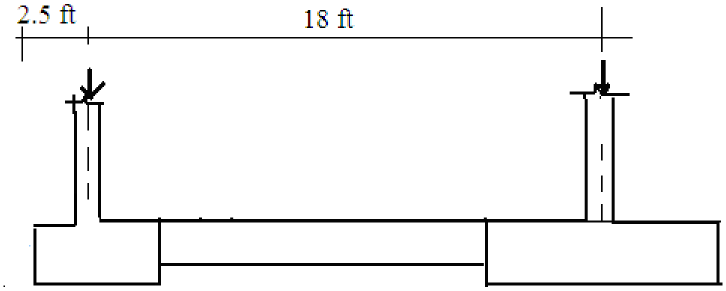
Computer plot V and M

Problem 7.8

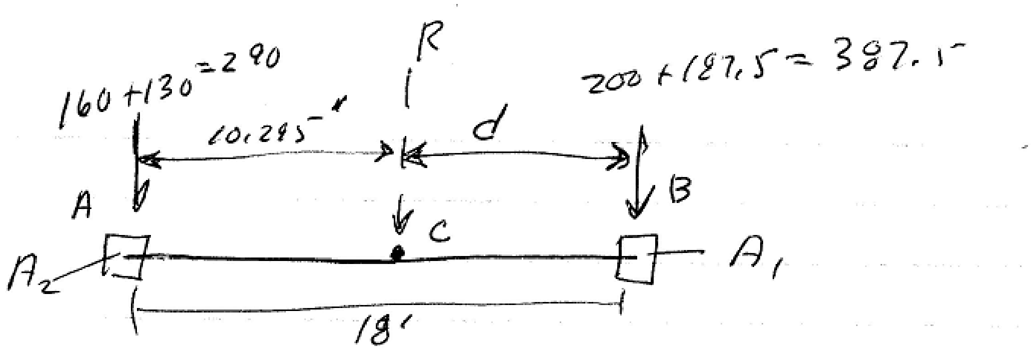
Dimension a strap footing for the situation shown. The exterior column A is 14 inches x 14 inches and carries a dead load of 160 kips and a live load of 130 kips; the interior column B is 18 inches x 18 inches and carries a dead load of 200 kips and a live load of 187.5 kips; the distance between the center lines of the columns is 18 ft. Assume the strap is placed such that it does not bear directly on the soil. Take the effective soil pressure as $q_e = 4.5\ \text{k/ft}^2$. Draw shear and moment diagrams using a factored load of $P_u = 1.2\ P_D + 1.6\ P_{L.}$



Plan



Elevation



$$R = 290 + 387.5 = 677.5$$

$$d = \frac{290(18)}{677.5} = 7.705$$

Locate centroid at C

$$A_1 + A_2 = \frac{677.5}{4.5} = 150.6\ ft^2$$

$$d\ A_1 = (10.295) A_2$$

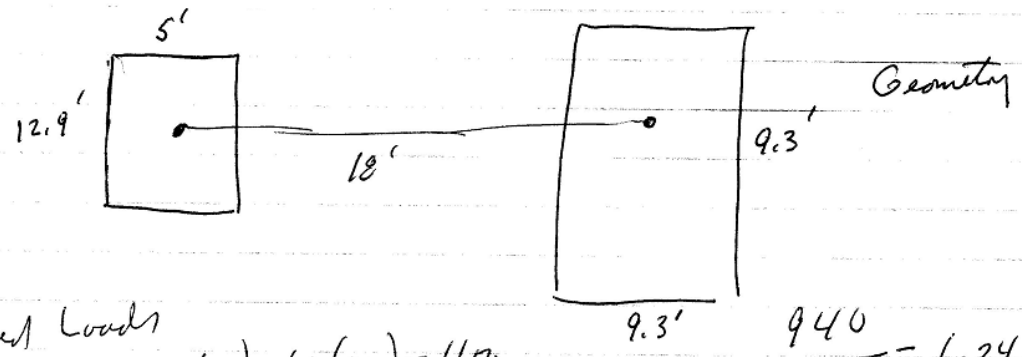
$\therefore$

$$A_1 = \frac{10.295}{7.705} A_2 = 1.336\ A_2$$

and

$$(1 + 1.336) A_2 = 150.6$$

$$\left.\begin{aligned} A_2 &= 64.5\ ft^2 \\ A_1 &= 150.6 - 64.5 = 86.1\ ft^2 \end{aligned}\right\}\ \text{ans}$$



Factored Loads

$$P_A = 1.2(160) + 1.6(130) = 400$$

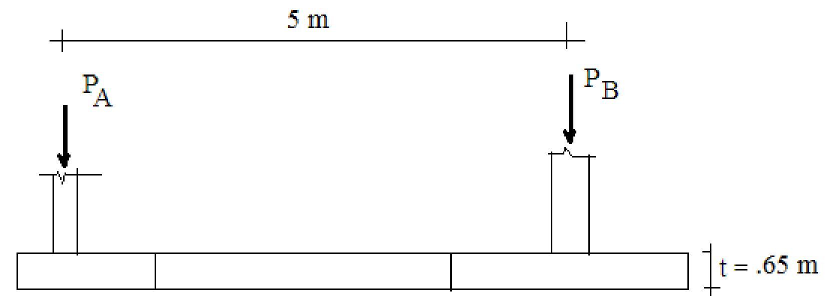
$$P_B = 1.2(200) + 1.6(187.5) = 540$$

$$q = \frac{940}{150.6} = 6.24\ k/ft^2$$

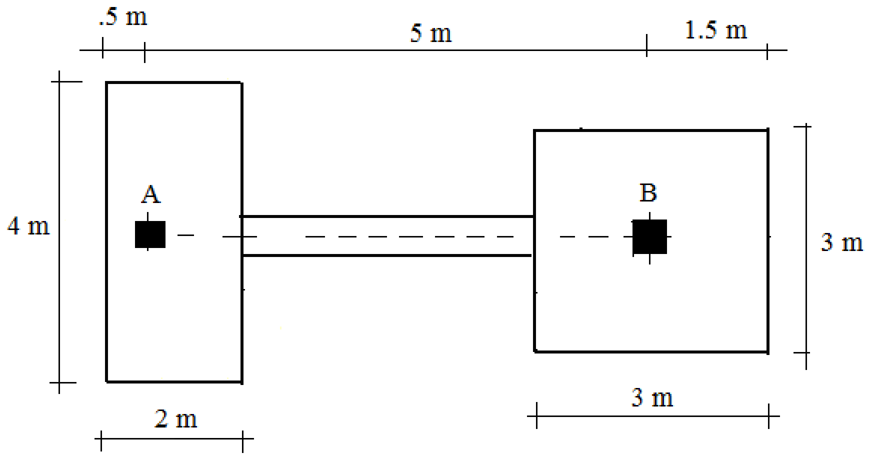
Based on "assumed" uniform pressure $\boxed{d = 7.66}$ vs 7.7

Problem 7.9

Column A is 350 mm x 350 mm and carries a dead load of 1300 kN and a live load of 450 kN. Column B is 450 mm x 450 mm and carries a dead load of 1400 kN and a live load of 800 kN. A strap footing is used to support the columns and the center line of Column A is 0.5 meter from the property line. Assume the strap is placed such that it does not bear directly on the soil. Determine the soil pressure distribution and the shear and bending moment distributions along the longitudinal direction corresponding to the factored loading, $P_u = 1.2\ P_D + 1.6\ P_L$.

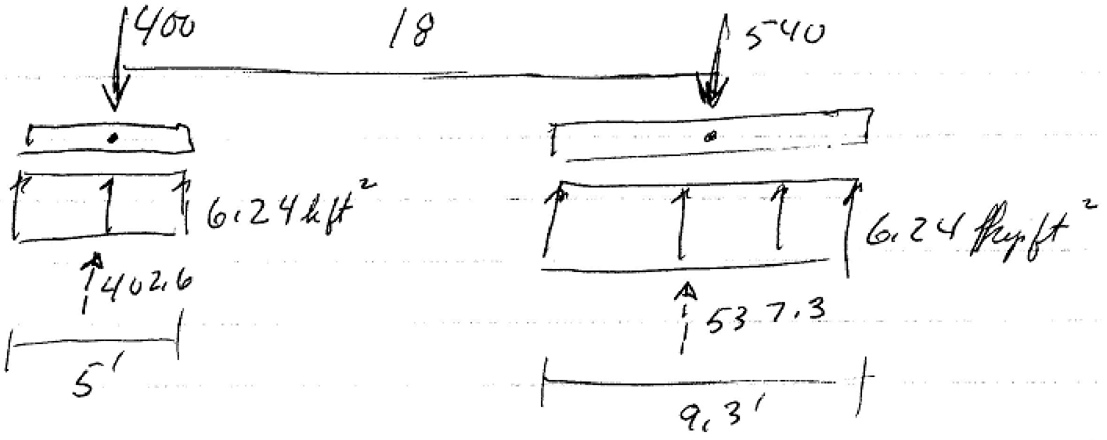


Elevation

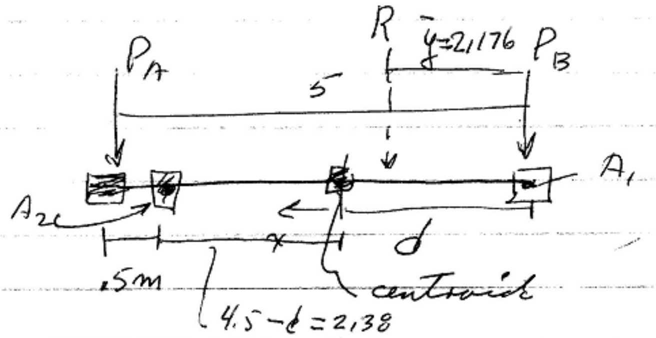


Plan

Factored Loads -

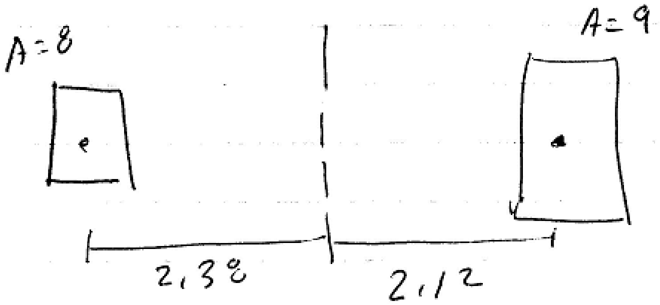


Essentially no moment in tie beam.



$A_1 = 9$
$A_2 = 8$ } Locate centroid

$$\frac{\cancel{8}A_2\cancel{(8)}(4.5)}{A_1+A_2} = d = \frac{8(4.5)}{17} = 2.12 \text{ m}$$



Neglect $\frac{bh^3}{12}$ term for each element

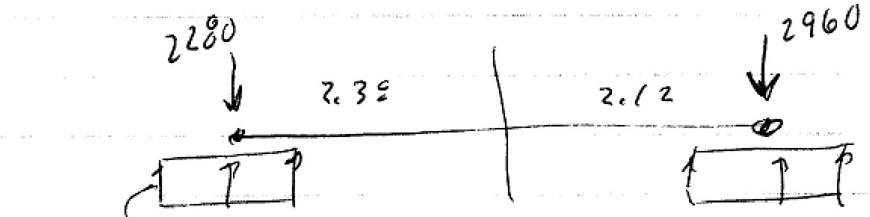
$$I \approx 8(2.38)^2 + 9(2.12)^2$$

$$\simeq 45.315 + 40.45 + \frac{4(2)^3}{12} + \frac{3(3)^3}{12}$$

$$\underbrace{+\frac{8}{3} + \frac{27}{4}}_{\text{small}}$$

$I \approx 85.765$

$q = 308.2 + \frac{293.4}{85.765} x \approx 308.2 + 3.42x$



$\approx 308.2 + 3.42(2.38)$

$\approx 316.$ 12528

$308.2 + 3.42(-2.12) = 301$

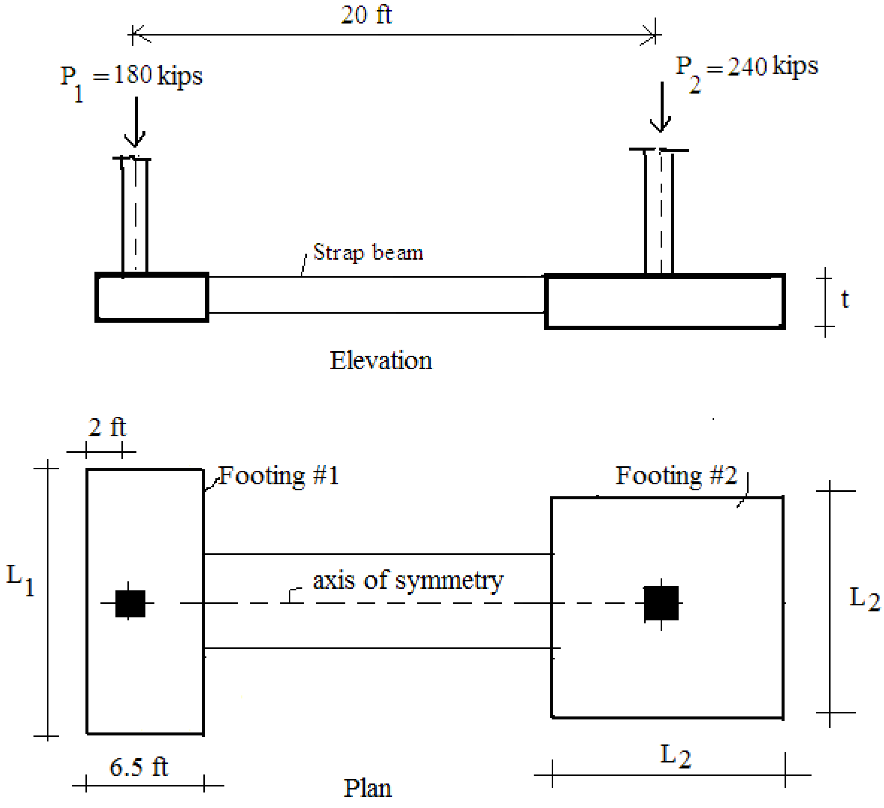
12709

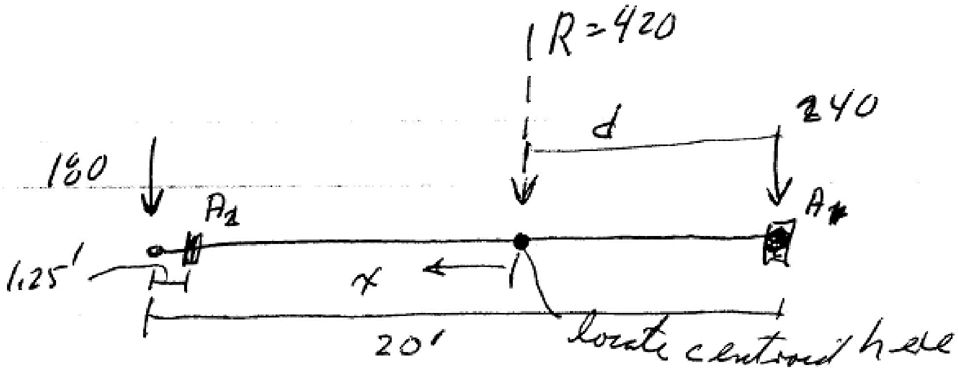
Problem 7.10

An exterior 18 in x18 in column with a total vertical service load of P_1 = 180 kips and an interior 20 in x20 in column with a total vertical service load of P_2 = 240 kips are to be supported at each column by a pad footing connected by a strap beam. Assume the strap is placed such that it does not bear directly on the soil.

(a) Determine the dimensions L_1 and L_2 for the pad footings that will result in a uniform effective soil pressure not exceeding 3 k/ft^2 under each pad footing. Use ¼ ft increments.

(b) Determine the soil pressure profile under the footings determined in part (a) when an additional loading, consisting of an uplift force of 80 kips at the exterior column and an uplift force of 25 kips at the interior column, is applied.





Resultant

$$R \cdot d = 180(20)$$

$$d = \frac{3600}{420} = 8.571$$

Centroid

$$A_1 + A_2 = \frac{420}{3} = 140\ ft^2$$

$$dA_1 = (\cancel{20 - 1.25 - d})A_2 \quad (20 - 1.25 - d)A_2$$

$$A_1 = \frac{\overset{10.18}{\cancel{9.921}}A_2}{8.571} = \overset{1.19}{\cancel{1.2}}A_2$$

$$(\overset{2.19}{\cancel{2.2}})A_2 = 140$$

$$A_2 = \overset{63.9}{\cancel{63.6}}\ ft^2$$

$$A_1 = 76.\overset{07}{\cancel{36}}\ ft^2$$

$$L_1 = \frac{\overset{63.9}{\cancel{63.6}}}{6.5} = \overset{9.83}{\cancel{9.79}}$$

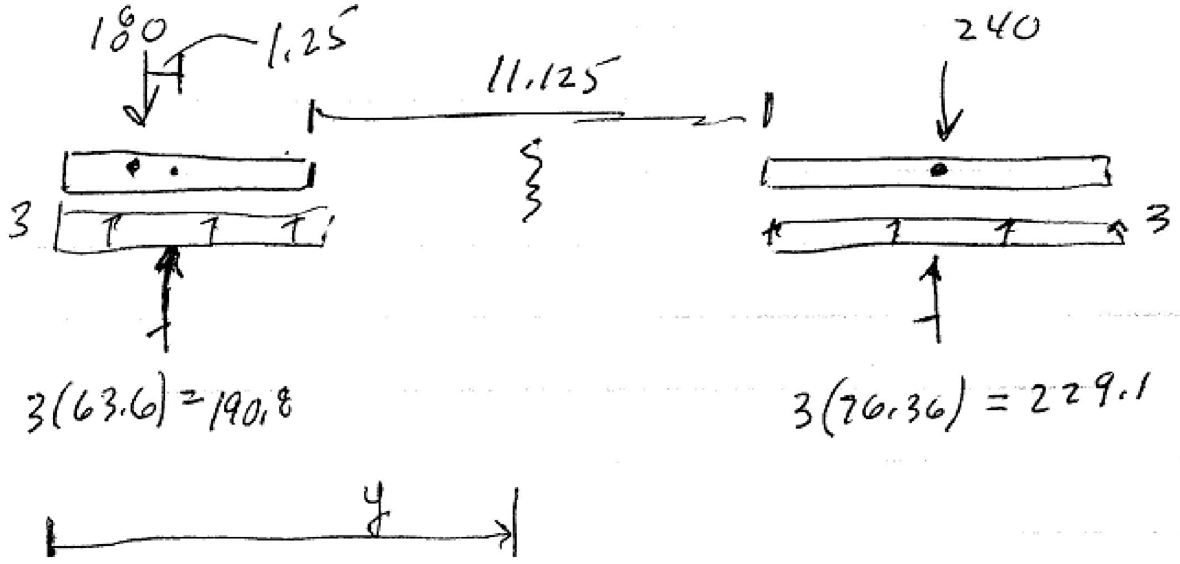
$$L_2 = \cancel{8.74}\ 8.72$$

Assume centroid location does not change

$$\boxed{\text{Take } L_1 = 9.75' \qquad L_2 = 8.75'}$$

Use original areas.

$$q = \frac{R}{A_1 + A_2} = \frac{420}{140} = 3\ k/'^2$$



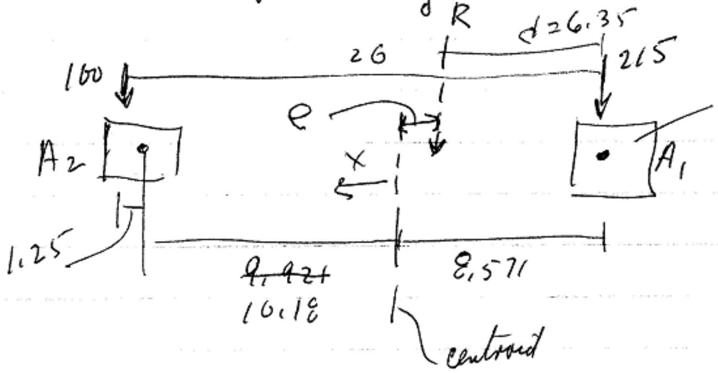
$11.125 > y > 6.5'$

$$M(y) = 190.8(y-3.25) - 180(y-2)$$

$$= 10.8y - 620.1 + 360$$

$$M(y) = 10.8y - 260.1$$ ans

Part b



$R = 315$

$R \cdot d = 100(20)$

$d = 6.35'$

$e = 8.571 - 6.35 = 2.22'$ (negative eccent)

$$q = \frac{315}{140} + \frac{315(-2.22)}{I}x$$

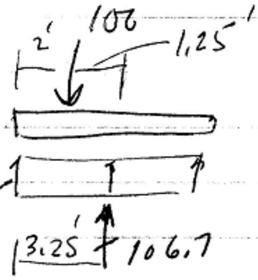
$$q = 2.25 - \frac{699.3}{I}x$$

$$I \approx A_1(8.571)^2 + A_2(10.18)^2$$

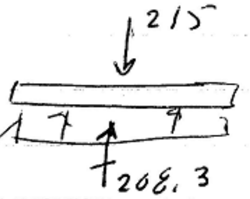
$$\approx (63.9)(10.18)^2 + 76.07(8.571)^2 =$$

$$\approx 6622 + 5588 \approx 12210$$

$$\boxed{q \approx 2.25 - .057y}$$



$2.25 - .057(10.18) = 1.67$



$2.25 - .057(-8.571) = 2.74$

$$M(y) \approx 106.7(y - 3.25) - 100(y - 2)$$

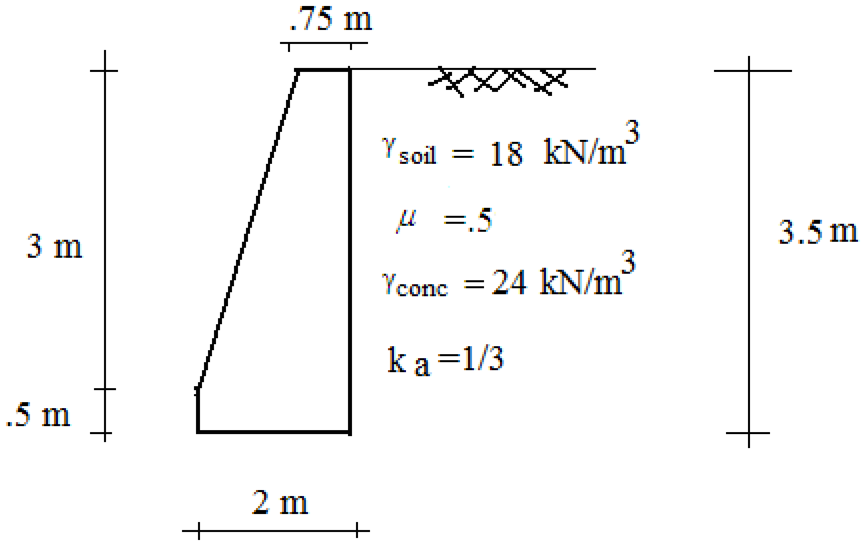
$$\approx 6.7y - 146.8$$

for ~~$y > 6.5$~~

$6.5 < y < 11.125$

Problem 8.1

For the concrete retaing wall shown, determine the factors of safety against sliding and overturning and the base pressure distribution.



$$P_a = \frac{1}{2}\gamma_{soil} H^2 k_a = \frac{1}{2}(18)(3.5)^2(\frac{1}{3}) = 49 \text{ kN}$$

$$M_{overturning} = P_a (\frac{H}{3}) = 49(\frac{3.5}{3}) = 57.16 \text{ kN-m}$$

$$W_1 = (24)(0.75)(3)(1) = 54 \text{ kN}$$

$$W_2 = (24)(\frac{3}{2})(1.25)(1) = 45 \text{ kN}$$

$$W_3 = (24)(2)(0.5)(1) = 24 \text{ kN}$$

$$N = N = \sum W_i = 123 \text{ kN}$$

$$F_{max} = \mu N = .5(123) = 61.5 \text{ kN}$$

$$F.S._{sliding} = \frac{F_{max}}{P_a} = \frac{61.5}{49} = 1.25$$

$$M_{resisting} = W_1(1.625) + W_2(\frac{2}{3})(1.25) + W_3(1) = 149.2 \text{ kN-m}$$

$$F.S._{overturning} = \frac{M_{resisting}}{M_{overturning}} = \frac{149.2}{57.16} = 2.6$$

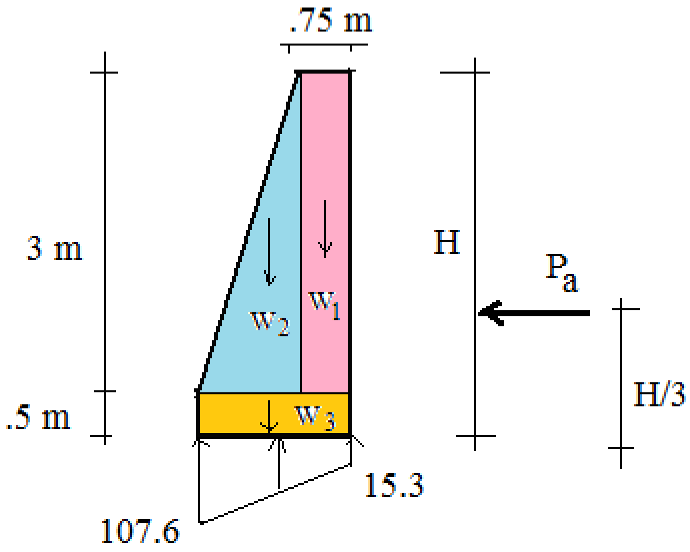
$$M_{net} = M_{resisting} - M_{overturning} = 92 \text{ kN-m}$$

$$\bar{x} = \frac{M_{net}}{N} = .748$$

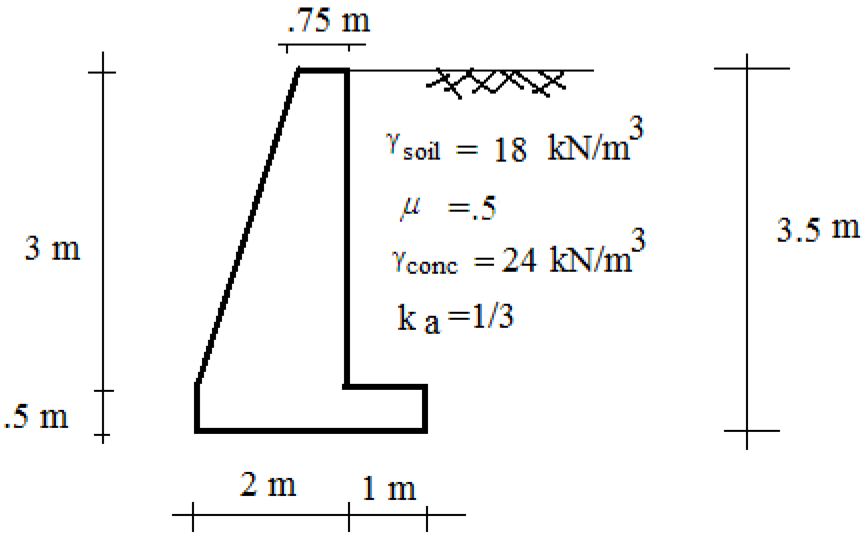
$$e = \frac{L}{2} - \bar{x} = .25 \text{ m}$$

$$q_1 = \frac{N}{L}\left\{1 + \frac{6\,e}{L}\right\} = \frac{123}{2}\left(1 + \frac{6(.25)}{2}\right) = 107.6 \text{ kN/m}^2$$

$$q_2 = \frac{N}{L}\left\{1 - \frac{6\,e}{L}\right\} = \frac{123}{2}\left(1 - \frac{6(.25)}{2}\right) = 15.3 \text{ kN/m}^2$$



Problem 8.2
For the concrete retaing wall shown, determine the factors of safety against sliding and overturning and the base pressure distribution. Use the Rankine theory for soil pressure computations.



$$P_a = \frac{1}{2}\gamma_{soil}\ H^2\ k_a = \frac{1}{2}(18)(3.5)^2(\frac{1}{3}) = 49\ \text{kN}$$

$$M_{overturning} = P_a\ (\frac{H}{3}) = 49(\frac{3.5}{3}) = 57.16\ \text{kN-m}$$

$$W_1 = (24)(0.75)(3)(1) = 54\ \text{kN}$$

$$W_2 = (24)(\frac{3}{2})(1.25)(1) = 45\ \text{kN}$$

$$W_3 = (24)(3)(0.5) = 36\ \text{kN}$$

$$W_4 = (18)(3)(1) = 24\ \text{kN}$$

$$N = \sum W_i = 189\ \text{kN}$$

$$F_{max} = \mu\ N = .5(123) = 94.5\ \text{kN}$$

$$\text{F.S.}_{\ sliding} = \frac{F_{max}}{P_a} = \frac{94.5}{49} = 1.928$$

$$M_{resisting} = 314.23\ \text{kN-m}$$

$$\text{F.S.}_{overturning} = \frac{M_{resisting}}{M_{overturning}} = \frac{314.23}{57.16} = 5.49$$

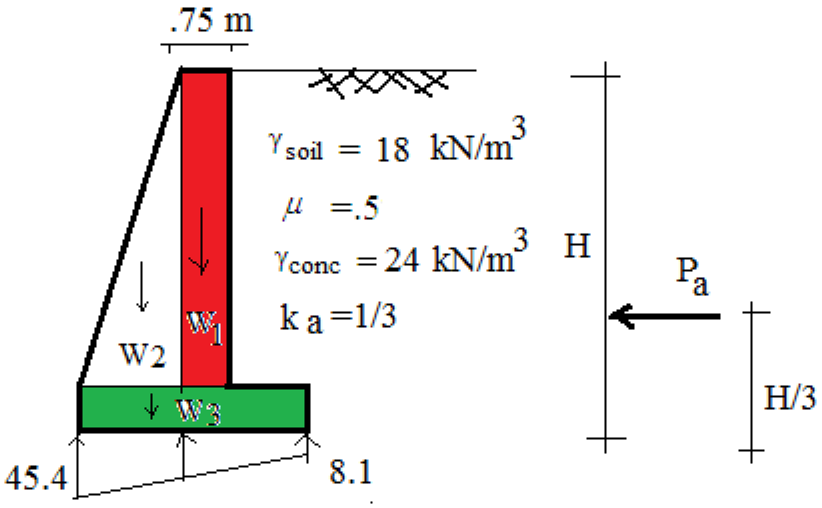
$$M_{net} = M_{resisting} - M_{overturning} = 257\ \text{kN-m}$$

$$\overline{x} = \frac{M_{net}}{N} = 1.36\ \text{m}$$

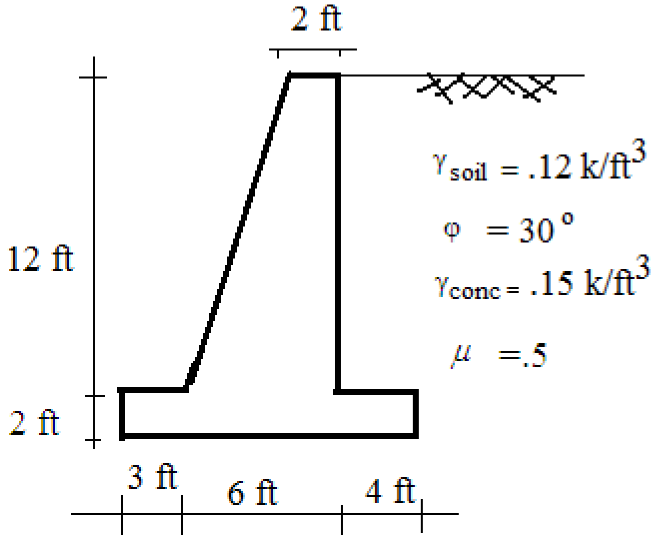
$$e = \frac{L}{2} - \overline{x} = 1.5 - 1.36 = .14\ \text{m}$$

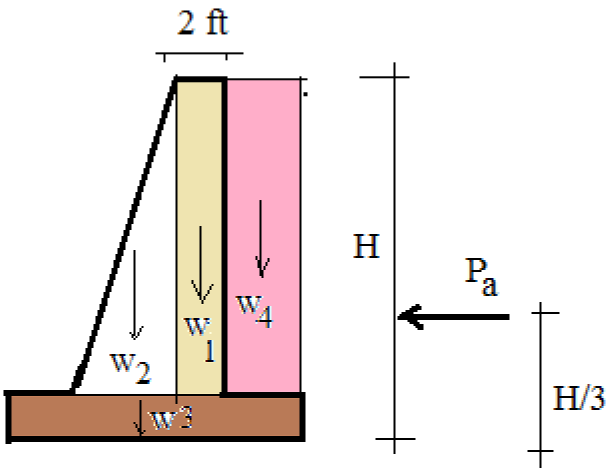
$$q_1 = \frac{N}{L}\left\{1 + \frac{6\ e}{L}\right\} = 45.4\ \text{kN/m}^2$$

$$q_2 = \frac{N}{L}\left\{1 - \frac{6\ e}{L}\right\} = 8.1\ \text{kN/m}^2$$



Problem 8.3
For the concrete retaining wall shown, determine the factors of safety against sliding and overturning and the base pressure distribution. Use the Rankine theory for soil pressure computations





$$P_a = \frac{1}{2} k_a \gamma_s H^2 = \frac{1}{2}(\frac{1}{3})(.12)(14)^2 = 3.92 \text{ kip}$$

$$W_1 = .15\ (2)\ (12) = 3.6 \text{ kip}$$

$$W_2 = .15\ (\frac{1}{2})(4\ )(12) = 3.6 \text{ kip}$$

$$W_3 = .15\ (2)\ (13) = 3.9 \text{ kip}$$

$$W_4 = .12\ (4)\ (12) = 5.76 \text{ kip}$$

The normal and horizontal forces are

$$N = W_1 + W_2 + W_3 + W_4 = 16.86 \text{ kip}$$

$$F_{max} = \mu N = .5\ (16.86) = 8.43 \text{ kips}$$

Next we compute the factors of safety.

$$F.S._{sliding} = \frac{F_{max}}{P_a} = \frac{8.43}{3.92} = 2.15$$

$$M_{B_{overturning}} = P_a(\frac{H}{3}) = 3.92(\frac{14}{3}) = 18.24 \text{ k-ft}$$

$$M_{B_{resisting}} = W_1(8)+W_2(2.67)+W_3(6.5)+W_4(11)$$

$$= 3.6\ (8)+3.6\ (2.67)+3.9(6.5)+5.76(11) = 127.1 \text{ k-ft}$$

$$F.S._{overturning} = \frac{M_{B_{resisting}}}{M_{B_{overturning}}} = \frac{127.1}{18.24} = 6.97$$

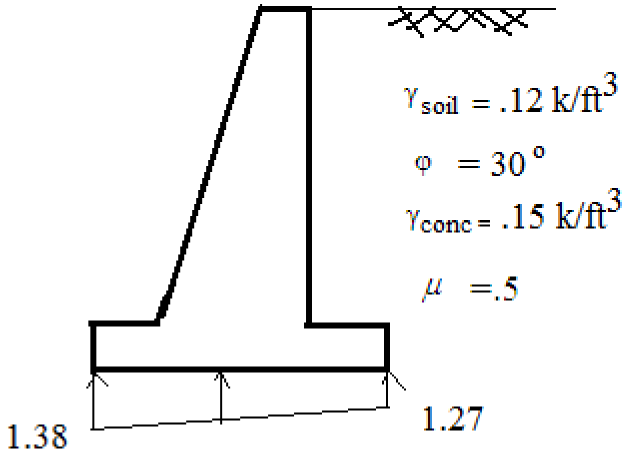
$$M_{Bnet} = M_{B_{resisting}} - M_{B_{overturning}} = 108.86 \text{ k-ft}$$

$$\bar{x} = \frac{M_{Bnet}}{N} = \frac{108.86}{16.86} = 6.45\text{ft}$$

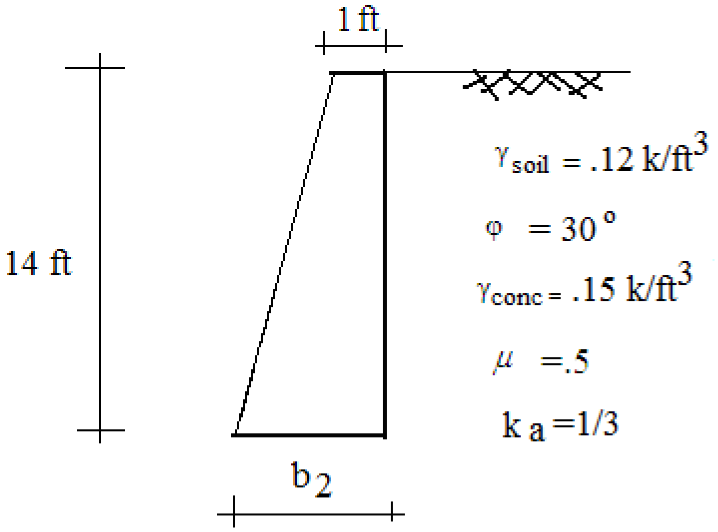
$$e = \frac{L}{2} - \bar{x} = \frac{13}{2} - 6.45 = .05 \text{ ft}$$

Using the above values, the peak pressures are

$$q = \frac{N}{L}(1 \pm \frac{6\,e}{L}) = \frac{16.86}{13}(1 \pm \frac{6(.05)}{13}) \quad \Rightarrow q_1 = 1.38 \text{ kip/ft}^2 \; q_2 = 1.27 \text{ kip/ft}^2$$



Problem 8.4



$$\text{F.S.}_{\text{ sliding}} = \left(\frac{\mu\ \gamma_c}{k_a\ \gamma_s}\right)\ \left(\frac{b_2}{H}\right)\ \left(1+\frac{b_1}{b_2}\right)$$

$$1.25 = \frac{.5\ (.15)}{(1/3)\ (.12)}\ \left(\frac{b_2}{14}\right)\ \left(1+\frac{1}{b_2}\right) \qquad \Rightarrow \qquad b_2 = 8.33$$

$$\text{F.S.}_{\text{overturning}} = \frac{2\ \gamma_c}{k_a\ \gamma_s}\left(\frac{b_2}{H}\right)^2 \left\{1-\frac{1}{2}\left(\frac{b_1}{b_2}\right)^2+\frac{b_1}{b_2}\right\}$$

$$\text{F.S.}_{\text{ sliding}} = \left(\frac{\mu\ \gamma_c}{k_a\ \gamma_s}\right)\ \left(\frac{b_2}{H}\right)\ \left(1+\frac{b_1}{b_2}\right)$$

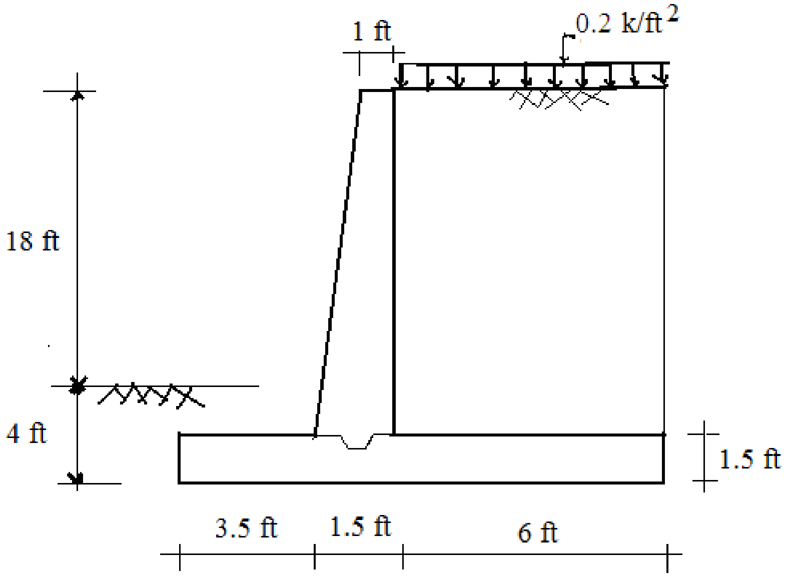
$$\text{F.S.}_{\text{overturning}} = \frac{2(.15)}{(1/3)\ (.12)}\left(\frac{b_2}{14}\right)^2 \left\{1-\frac{1}{2}\left(\frac{1}{b_2}\right)^2+\frac{1}{b_2}\right\} \geq 1.75$$

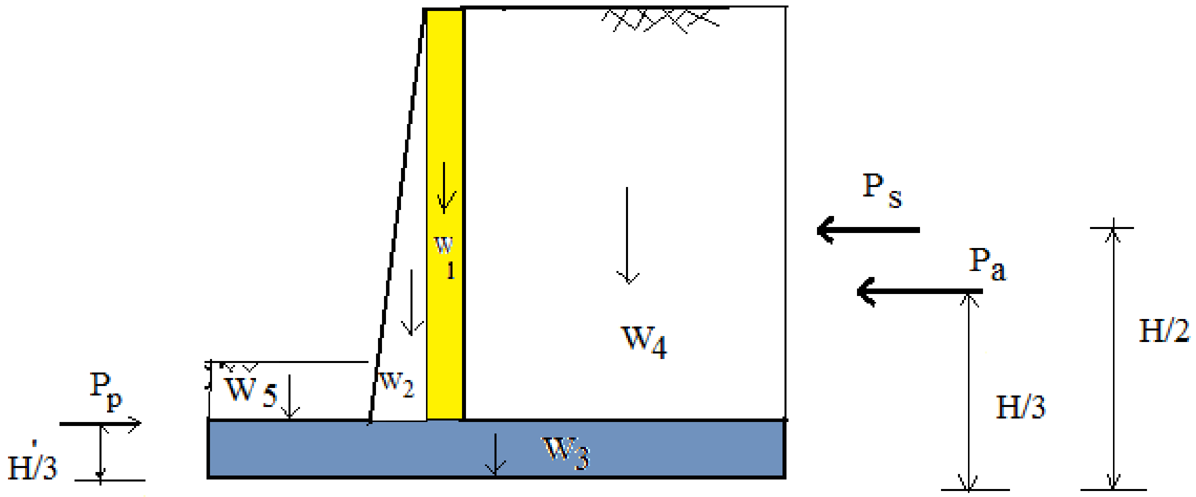
For $b_2 = 8.33$

$$\text{F.S.}_{\text{overturning}} = \frac{2(.15)}{(1/3)\ (.12)}\left(\frac{8.33}{14}\right)^2 \left\{1-\frac{1}{2}\left(\frac{1}{8.33}\right)^2+\frac{1}{8.33}\right\} = 2.95 \geq 1.75$$

Problem 8.5

Assume: $\gamma_{\text{soil}} = 0.12\ \text{k/ft}^3$, $\gamma_{\text{concrete}} = 0.15\ \text{k/ft}^3$, μ= .5 and Φ = 30^0





$$P_a = \frac{1}{2}\gamma_{soil}\ H^2\ k_a = \frac{1}{2}(.12)(22)^2(\frac{1}{3}) = 9.68\ \text{kip}$$

$$P_s\ =\ k_a\ w_s\ H\ =\ (\frac{1}{3})\ (.2)\ (22) = 1.47$$

$$P_p\ =\ \frac{1}{2}\ k_p\ \gamma_s H'^2 = \frac{1}{2}(3)(.12)(4)^2 = 2.88\ \text{kip}$$

$$\sum F_{horizontal} = P_a + P_s\ -\ P_p = 8.27\ \text{kip}$$

$$M_{\ overturning}\ = P_a\ (\frac{H}{3}) + P_s(\frac{H}{2}) = 9.68\ (\frac{22}{3}) + 1.47(\frac{22}{2}) = 87.15$$

$$W_1\ =\ .15\ (1)\ (20.5)\ =\ 3.075$$

$$W_2\ =\ .15\ (.5)\ (\frac{20.5}{2})\ =\ .77$$

$$W_3\ =\ .15\ (b11)\ (1.5)\ =\ 2.475$$

$$W_4\ =\ .12\ (6)\ (20.5)\ =\ 14.76$$

$$W_5\ =\ .12\ (2.5)\ (3.5)\ =\ 1.05$$

$$N\ =\ \sum W_i = 22$$

$$F_{max} = \mu\ N\ = 11$$

$$F.S._{sliding}\ =\ \frac{F_{max}}{\sum F_{horizontal}} = \frac{11}{8.27} = 1.33$$

$$M_{B_{resisting}} = W_1(4.5)+W_2(3.83)+W_3(5.5)+W_4(8)+W_5(1.75)+P_p(1.33)$$

$$= 3.075(4.5)+.768(3.83)+2.475(5.5)+14.76(8)+1.05(1.75)+2.88(1.33) = 154 \text{ kip-ft}$$

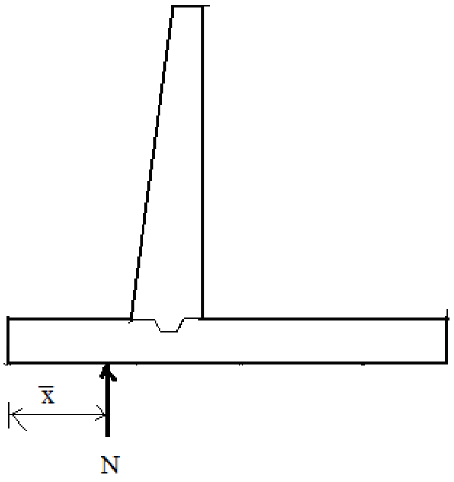
$$F.S._{overturning} = \frac{M_{B_{resisting}}}{M_{B_{overturning}}} = \frac{154}{87.15} = 1.76$$

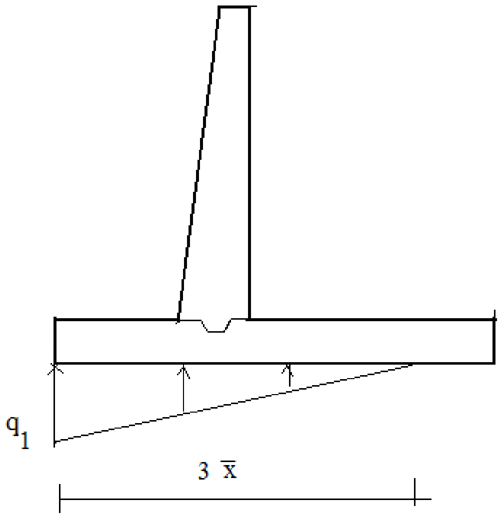
$$M_{Bnet} = M_{B_{resisting}} - M_{B_{overturning}} = 66.85 \text{ kip-ft}$$

$$\bar{x} = \frac{M_{Bnet}}{N} = \frac{66.85}{22} = 3\text{ft}$$

$$e = \frac{L}{2} - \bar{x} = \frac{11}{2} - 3 = 3.5 \text{ ft} \succ \frac{L}{6} = 1.83$$

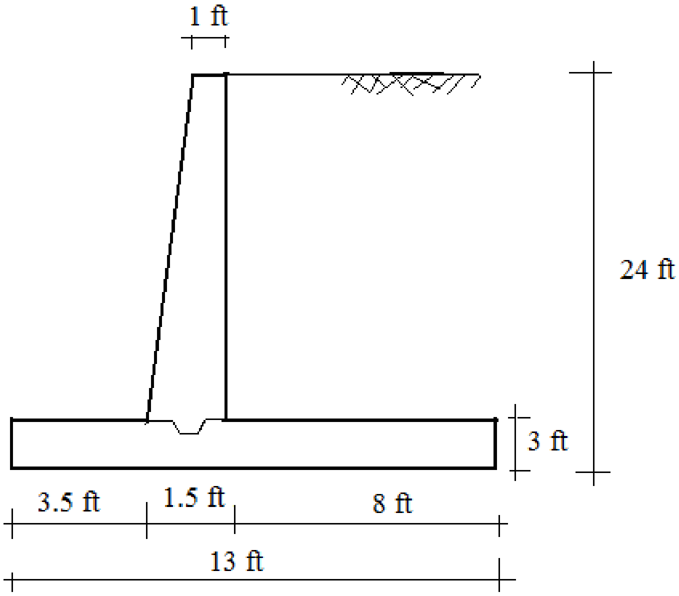
$$N = \frac{q_1(3a)}{2} \qquad \Rightarrow \qquad q_1 = \frac{(22)(2)}{3(3)} = 4.89 \text{ kip/ft}^2$$

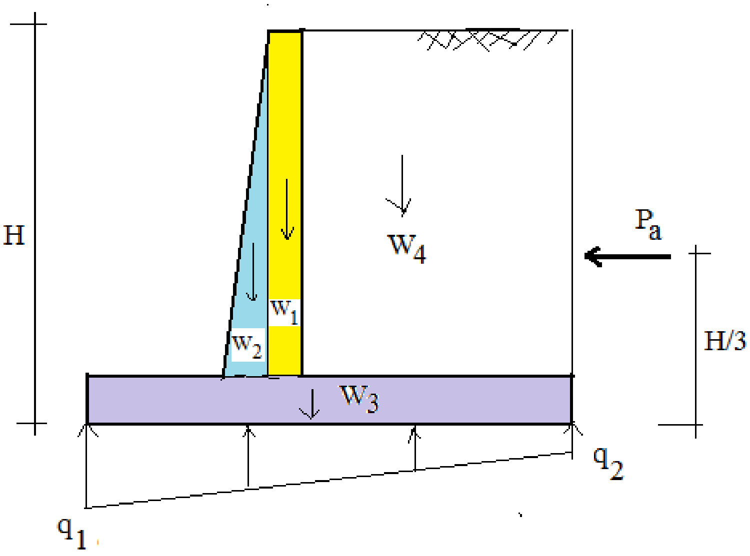




Problem 8.6

Allowable soil pressure = 5.0 ksf , γ_{soil}=.12 k/ft^3 and $\gamma_{concrete}$=.15 k/ft^3





$$P_a = \frac{1}{2} k_a \gamma_s H^2 = \frac{1}{2}(\frac{1}{3})(.12)(24)^2 = 11.52 \text{ kip}$$

$$M_{B_{overturning}} = P_a(\frac{H}{3}) = 11.52(\frac{24}{3}) = 92.16 \text{ kip-ft}$$

$$W_1 = .15\ (1)\ (21) = 3.15 \text{ kip}$$

$$W_2 = .15\ (\frac{1}{2})(.5\ )(21) = .78 \text{ kip}$$

$$W_3 = .15\ (3)\ (13) = 5.85 \text{ kip}$$

$$W_4 = .12\ (8)\ (21) = 20.1 \text{ kip}$$

$$N = \sum W_i = 30 \text{ kip}$$

$$F_{max} = \mu\ N = .5\ (30) = 15 \text{ kips}$$

$$F.S._{sliding} = \frac{F_{max}}{P_a} = \frac{15}{11.52} = 1.3$$

$$M_{B_{resisting}} = W_1(4.5)+W_2(3.83)+W_3(6.5)+W_4(9)$$

$$= 3.15(4.5)+.78(3.83)+5.85(6.5)+20.1\ (9) = 236 \text{ kip-ft}$$

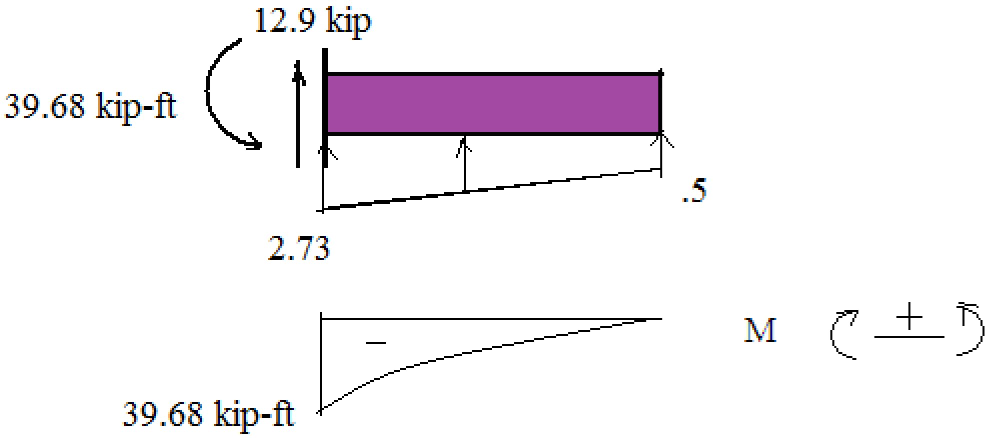
$$F.S._{overturning} = \frac{M_{B_{resisting}}}{M_{B_{overturning}}} = \frac{236}{92.16} = 2.56$$

$$M_{Bnet} = M_{B_{resisting}} - M_{B_{overturning}} = 143.84 \text{ kip-ft}$$

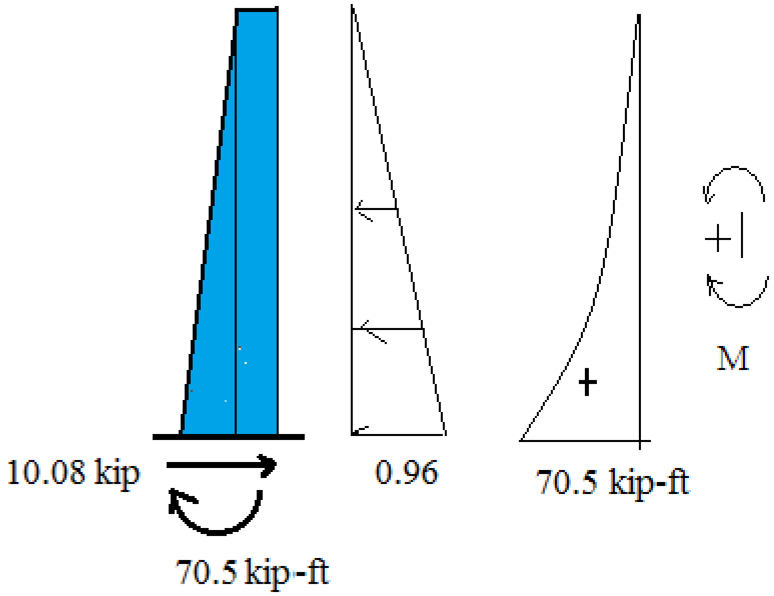
$$\overline{x} = \frac{M_{Bnet}}{N} = \frac{143.84}{30} = 4.79\text{ft}$$

$$e = \frac{L}{2} - \overline{x} = \frac{13}{2} - 4.79 = 1.7 \text{ ft}$$

$$q = \frac{N}{L}(1 \pm \frac{6\ e}{L}) = \frac{30}{13}(1 \pm \frac{6(1.7)}{13}) \quad \Rightarrow q_1 = 4.12 \text{ kip/ft}^2\ q_2 = 0.5 \text{ kip/ft}^2$$



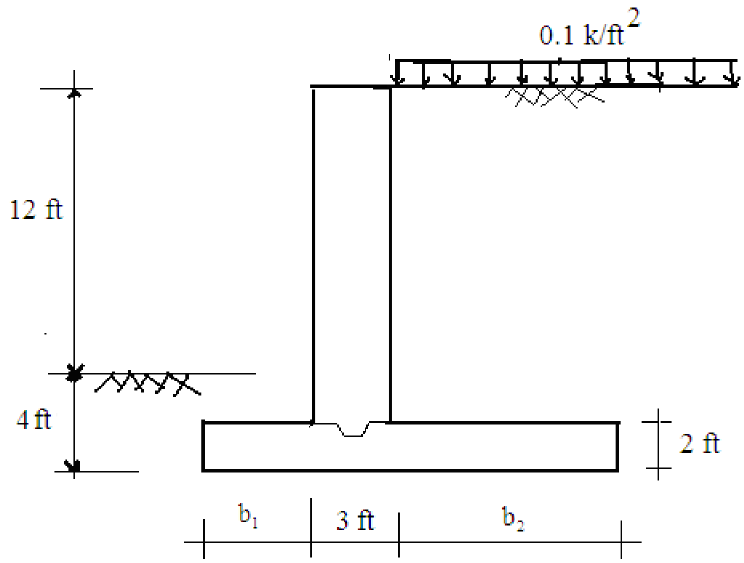
$p = k_a \gamma_s H = (\frac{1}{3})(.12)(24) = 0.96$ kip/ft



Problem 8.7

Suggest values for b_1 and b_2. Take the safety factors for sliding and over turning to be equal to 2.

Assume: γ_{soil}=.12 k/ft^3, $\gamma_{concrete}$=.15 k/ft^3, μ = .57, and Φ = 30^0



$$P_a = \frac{1}{2}\gamma_{soil}\ H^2\ k_a = \frac{1}{2}(.12)(16)^2(\frac{1}{3}) = 5.12\ \text{kip}$$

$$P_s\ =\ k_a\ w_s\ H\ =\ (\frac{1}{3})\ (.1)\ (16) = .53$$

$$P_p\ =\ \frac{1}{2}\ k_p\ \gamma_s H'^2 = \frac{1}{2}(3)(.12)(4)^2 = 2.88\ \text{kip}$$

$$\sum F_{horizontal} = P_a + P_s\ -\ P_p = 2.77$$

$$M_{\ overturning} = P_a\ (\frac{H}{3}) + P_s(\frac{H}{2}) = 5.12\ (\frac{16}{3}) + .53(\frac{16}{2}) = 31.55$$

$$W_1\ =\ .15\ (3)\ (14)\ =\ 6.3$$

$$W_2\ =\ .15\ (2)\ (b_1{+}b_2{+}3)\ =\ .3(b_1{+}b_2{+}3)$$

$$W_3\ =\ .12\ (b_2)\ (20)\ =\ 2.4b_2$$

$$W_4\ =\ .12\ (b_1)\ (2)\ =\ 2.4b_1$$

$$N\ =\ \sum W_i = 6.3{+}.3(b_1{+}b_2{+}3){+}2.4b_2 +\ 2.4b_1 = 7.2 + 2.7(b_1{+}b_2)$$

$$F_{max} = \mu\ N\ =\ .57\ (7.2 + 2.7(b_1{+}b_2))\ = 4.1{+}1.54(b_1{+}b_2)$$

$$F.S._{sliding}\ = \frac{F_{max}}{\sum F_{horizontal}} = \frac{4.1{+}1.54(b_1{+}b_2)}{2.77} = 2 \qquad \rightarrow \qquad b_1{+}b_2 = 0.935$$

$$\therefore \qquad b_2 = 0.935 - b_1$$

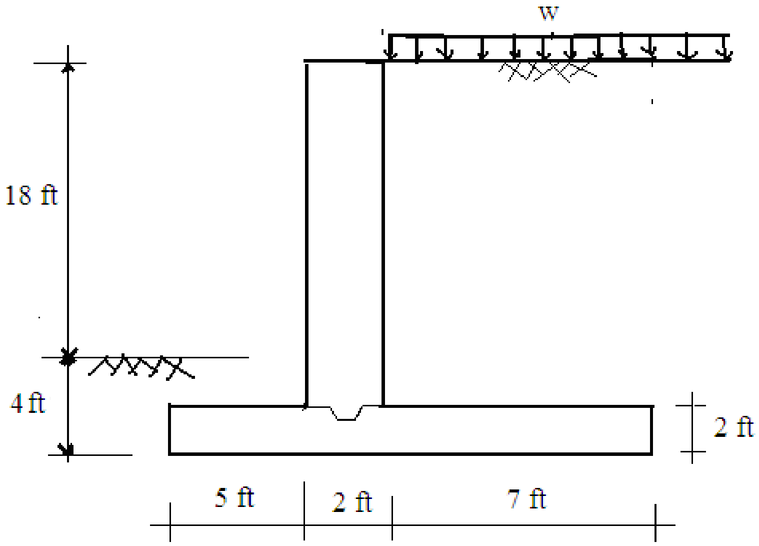
$$M_{B_{resisting}} = W_1(1.5+b_1)+W_2(.5b_1+.5b_2+1.5)+W_3(b_1+.5b_2+3)+W_4(2.5)+P_p(1.33) =$$

$$= 6.3(1.5+b_1)+.3(b_1+b_2+3)(.5b_1+.5b_2+1.5)+16.8(b_1+.5b_2+3)+2.4b_1(2.5)+2.88(1.33) =$$

$$= 106.4+3.9b_1$$

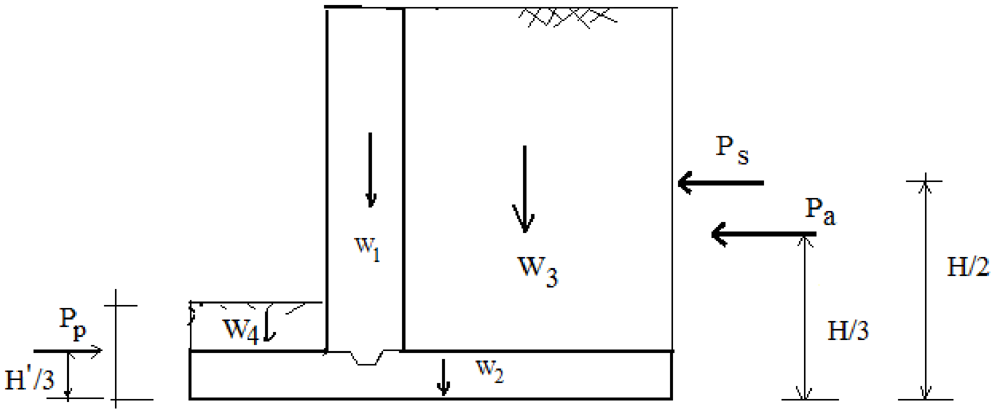
$$F.S._{overturning} = \frac{M_{B_{resisting}}}{M_{B_{overturning}}} = \frac{106.4+3.9b_1}{31.55} = 2 \qquad b_1 \le 0 \qquad \therefore b_2 = .935\text{ft}$$

Problem 8.8



Determine the minimum value of w at which soil failure occurs (i.e., the soil pressure exceeds the allowable soil pressure).

Assume: $q_{allowable} = 5\ \text{k/ft}^2$, $\gamma_{soil} = .12\ \text{k/ft}^3$, $\gamma_{concrete} = .15\ \text{k/ft}^3$, $\mu = .57$ and $\Phi = 30^0$



$$P_a = \frac{1}{2}\gamma_{soil}\ H^2\ k_a = \frac{1}{2}(.12)(22)^2(\frac{1}{3}) = 9.68\ \text{kip}$$

$$P_s\ =\ k_a\ w_s\ H\ =\ (\frac{1}{3})\ (w_s)\ (22) = 7.33\ w_s$$

$$P_p\ =\ \frac{1}{2}\ k_p\ \gamma_s (H^{'})^2 = \frac{1}{2}(3)(.12)(4)^2 = 2.88\ \text{kip}$$

$$\sum F_{horizontal} = P_a + P_s\ -\ P_p = 9.68 + 7.33\ w_s - 2.88 = 6.88 + 7.33\ w_s$$

$$M_{overturning} = P_a\ (\frac{H}{3}) + P_s(\frac{H}{2}) = 9.68(\frac{22}{3}) + 7.33\ w_s(11) = 71 + 80.6\ w_s$$

$$W_1\ =\ .15\ (2)\ (20)\ =\ 6\ \ \text{kip}$$

$$W_2\ =\ .15\ (2)\ (14)\ =\ 4.2\ \ \text{kip}$$

$$W_3\ =\ .12\ (7)\ (20)\ =\ 16.8\ \ \text{kip}$$

$$W_4\ =\ .12\ (5)\ (2)\ =\ 1.2\ \text{kip}$$

$$N\ =\ \sum W_i = 28.2\ \text{kip}$$

$$F_{max} = \mu\ N\ =\ .57\ (28.2)\ =\ 16\ \text{kip}$$

$$q = \frac{N}{L}(1\ +\ \frac{6\ e}{L}) \le 5 \qquad \frac{28.2}{14}(1\ +\ \frac{6\ e}{14}) \le 5 \qquad e = 3.46\ \text{ft}$$

$$\bar{x} = \frac{M_{net}}{N} = \frac{M_{net}}{28.2}$$

$$e = \frac{L}{2} - \bar{x} = 7 - \frac{M_{net}}{28.2} = 3.46 \qquad M_{net} = 99.8$$

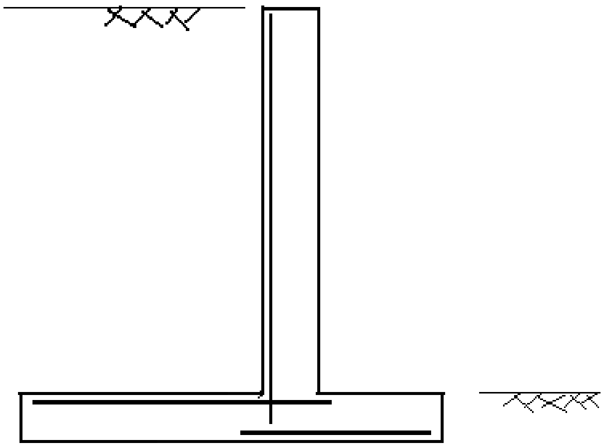
$$M_{B_{resisting}} = W_1(6) + W_2(7) + W_3(10.5) + W_4(2.5) + P_p(1.33) =$$

$$= 6(6) + 4.2(7) + 16.8(10.5) + 1.2(2.5) + 2.88(1.33) = 248.6$$

$$M_{net} = M_{resisting} - M_{overturning} \qquad 99.8 = 248.6 - (71 + 80.6\ w_s)$$

$$\therefore \qquad w_s = .96 \text{kip/ft}^2$$

Problem 8.9



(a)

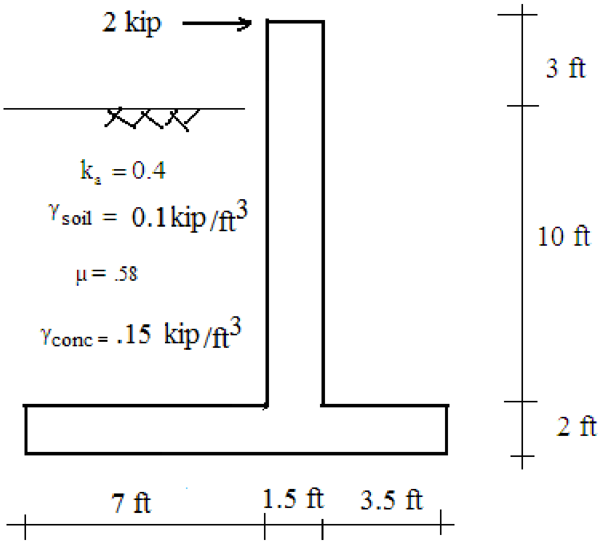
The answer is case (a).

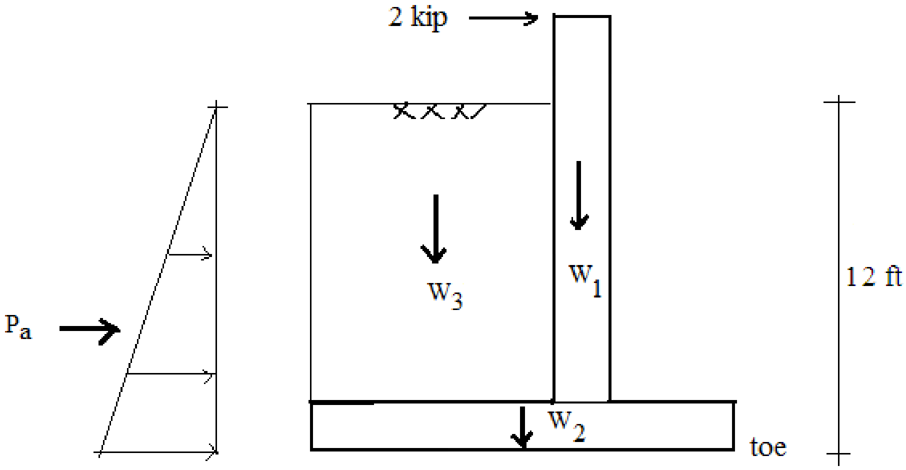
Problem 8.10

(a) Determine the factor of safety with respect to overturning and sliding.

(b) Identify the tension areas in the stem, toe, and heel and show the reinforcing pattern.

(c) Determine the location of the line of action of the resultant at the base of the footing





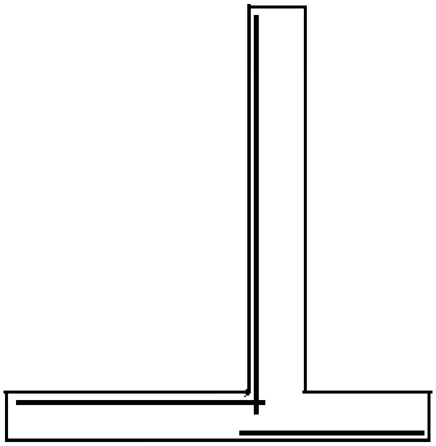
$$P_a = \frac{1}{2}\gamma_{soil}\ H^2\ k_a = \frac{1}{2}(.1)(12)^2(.4) = 2.88\ \text{kip}$$

$$M_{overturning} = P_a\ (\frac{H}{3}) + 2(15) = 2.88(\frac{12}{3}) + 30 = 41.52\ \text{kip-ft}$$

$$W_1 = (1.5)(13)(3)(.15) = 2.925\ \text{kip}$$
$$W_2 = (2)(12)(.15) = 3.6\ \text{kip}$$
$$W_3 = (7)(2)(010)(.1) = 7\ \text{kip}$$

$$N = \sum W = 13.525\ \text{kip}$$
$$F_{max} = \mu\ N = .58(13.525) = 7.84\ \text{kip}$$
$$F.S._{\ sliding} = \frac{F_{max}}{P_a} = \frac{7.84}{2.88} = 2.72$$

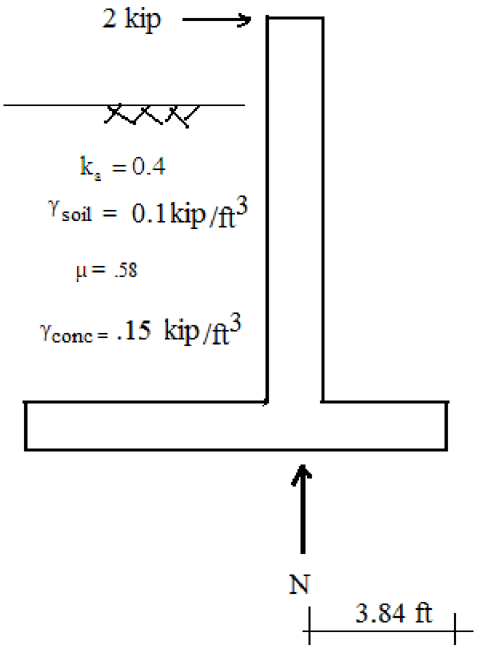
$$M_{resisting} = 93.53\ \text{kip-ft}$$
$$F.S._{overturning} = \frac{M_{resisting}}{M_{overturning}} = \frac{93.53}{41.52} = 2.25$$



$$M_{net} = M_{resisting} - M_{overturning} = 52 \text{ kip-ft}$$

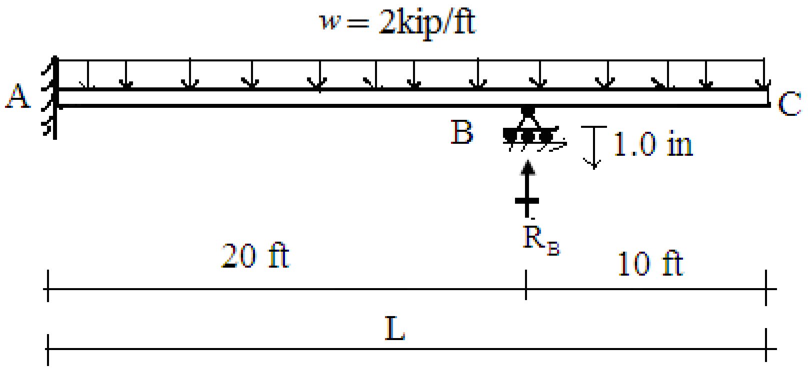
$$\bar{x} = \frac{M_{net}}{N} = \frac{52}{13.525} = 3.84$$

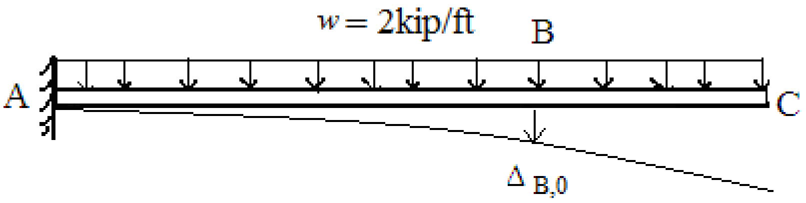
$$e = \frac{L}{2} - \bar{x} \;\; = 6 - 3.84 = 2.16 \;\text{ ft} \succ \frac{L}{6} = 2\text{ft}$$

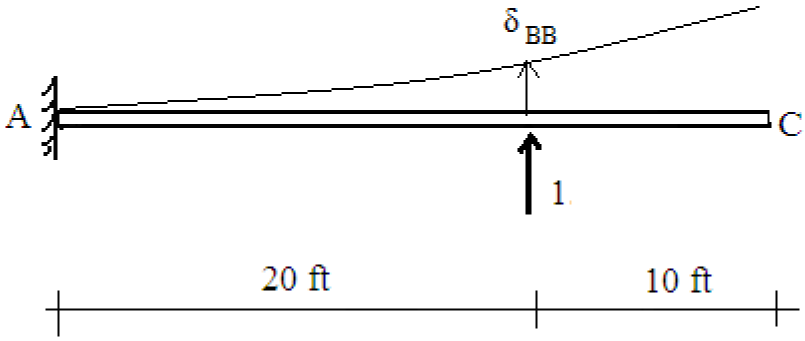


Problem 9.1

Take E=29,000 ksi and I= 200 in^4.







$$\begin{cases} \Delta_{B,0} = \dfrac{17wL^4}{243EI} = \dfrac{17(2)(30)^4(12)^3}{243(29000)(200)} = 33.78 \text{ in} \\ \delta_{BB} = \dfrac{8L^3}{81EI} = \dfrac{8(30)^3(12)^3}{81(29000)(200)} = 0.794 \text{ in} \end{cases}$$

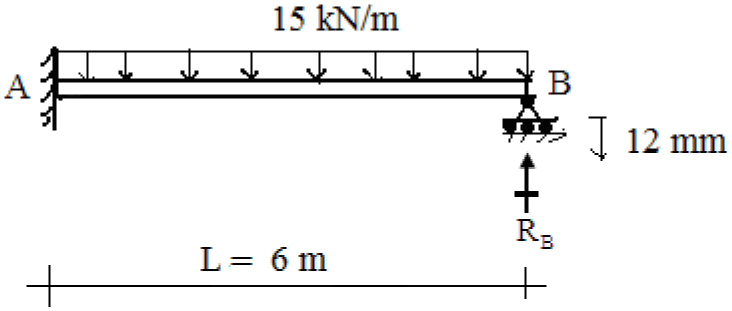
$$v_B = 1 \text{ in}$$

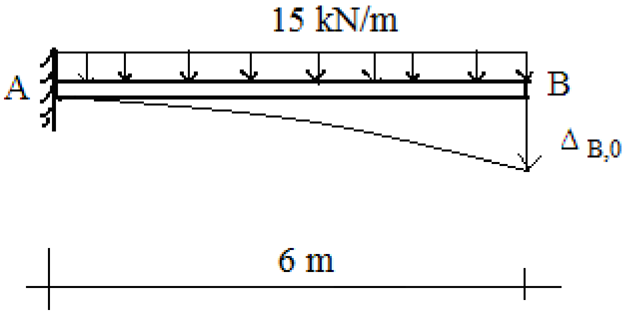
$$\uparrow^{+} \qquad -\Delta_{B,0} + R_B\ \delta_{BB} = -1\ \text{in}$$

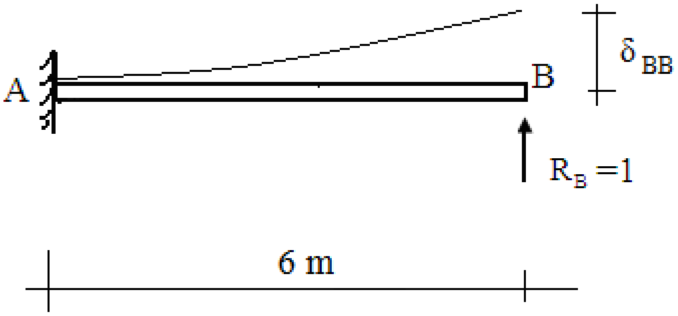
$$R_B = \frac{\Delta_{B,0} - 1}{\delta_{BB}} = \frac{33.78 - 1}{0.794} = 41.28\ \text{kip}\ \uparrow$$

Problem 9.2

Take E=200 GPa and I= $80(10)^6$ mm^4.







$$\uparrow^{+} \qquad -\Delta_{B,0} + R_B\ \delta_{BB} = -12\ \text{mm}$$

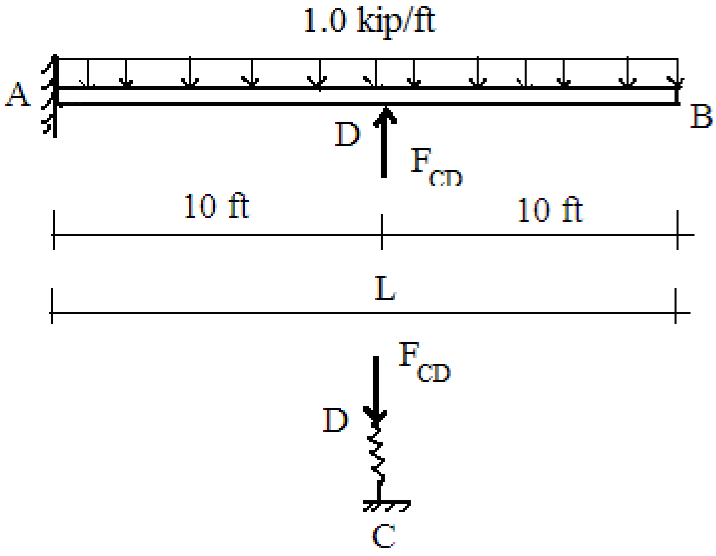
$$\nu_B = 12\ \text{mm}$$

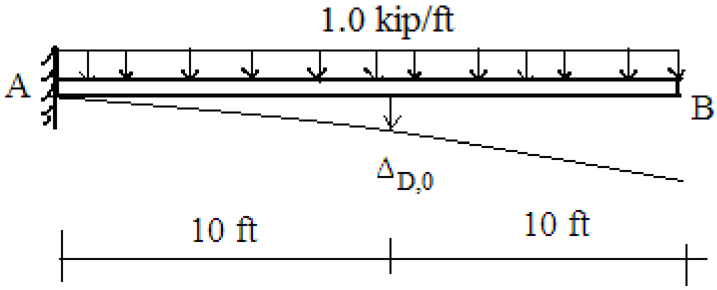
$$\textbf{Table 3.1}\begin{cases}\Delta_{B,0}=\dfrac{wL^4}{8EI}=\dfrac{15(6)^4(10)^9}{8(200)(80)10^6}=151.875\ \text{mm}\\[2ex] \delta_{BB}=\dfrac{L^{\ 3}}{3EI}=\dfrac{(6)^3(10)^9}{3(200)(80)10^6}=4.5\ \text{mm}\end{cases}$$

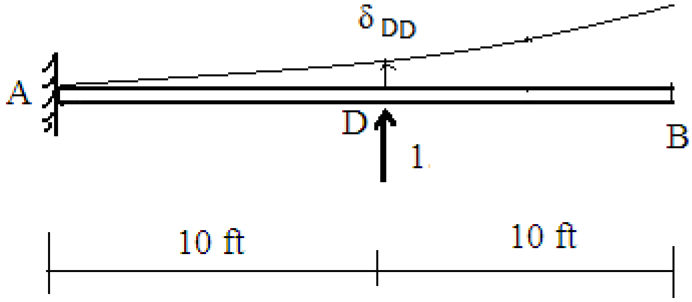
$$R_B=\frac{\Delta_{B,0}-12}{\delta_{BB}}=\frac{151.875-12}{4.5}=31.08\ \text{kN}\ \uparrow$$

Problem 9.3

kv= 60 k/in, E=29,000 ksi, I= 200 in^4







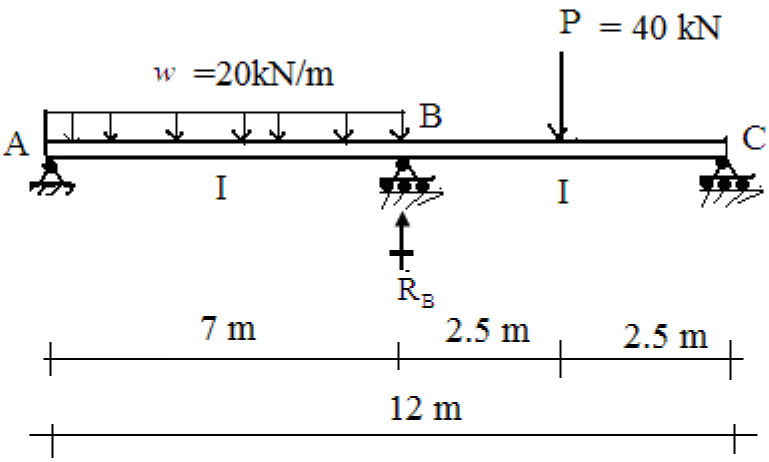
$$\begin{cases} \Delta_{D,0} = \dfrac{17wL^4}{384EI} = \dfrac{17(1.0)(20)^4(12)^3}{384(29000)(200)} = 2.11 \text{ in} \\ \delta_{DD} = \dfrac{L^{\ 3}}{24EI} = \dfrac{(20)^3(12)^3}{24(29000)(200)} = 0.0993 \text{ in} \end{cases}$$

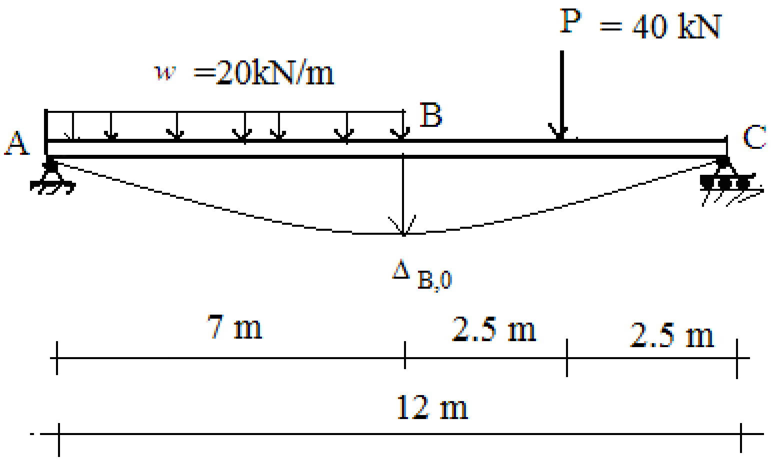
$$F_{CD} = \frac{\Delta_{D,0}}{\delta_{DD} + \dfrac{1}{k_V}} = \frac{2.11}{0.0993 + \dfrac{1}{60}} = 18.19 \text{ kip}$$

Problem 9.4

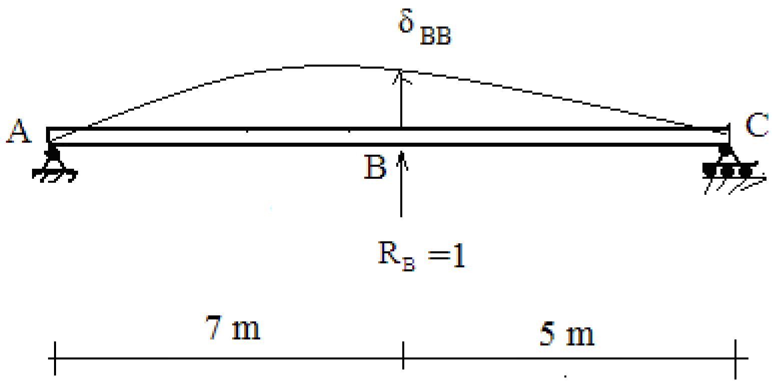
$L = 5$ m, $E = 200$ GPa, $I = 170(10)^6$ mm^4

(a)





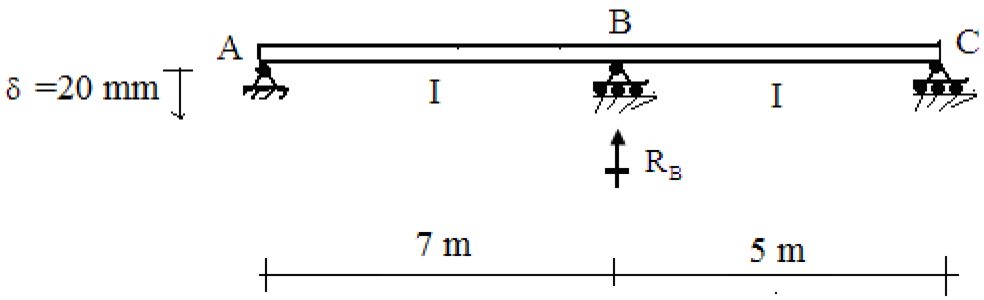
$$\Delta_{B,0} = 110.49 \text{ mm}$$

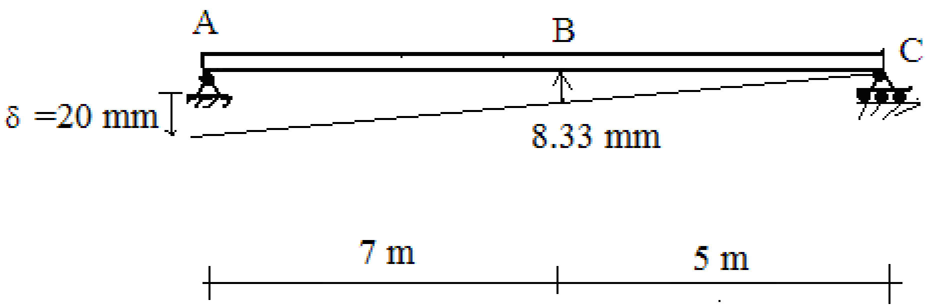


$$\delta_{BB} = \frac{L^3}{48EI} = \frac{(12)^3(10)^9}{48(200)(170)10^6} = 1.059 \text{ mm}$$

$$R_B = \frac{\Delta_{B,0}}{\delta_{BB}} == 110.4 \text{ kN} \uparrow$$

(b)

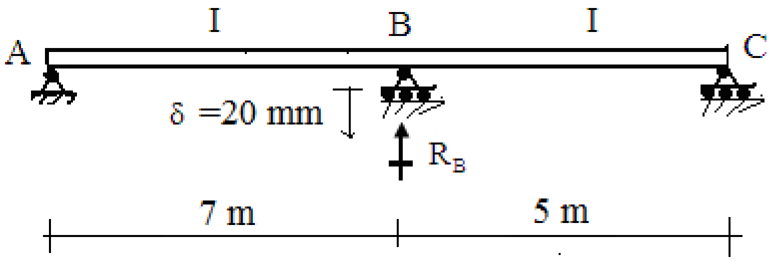




$$\delta_{BB} = \frac{L^3}{48EI} = \frac{(12)^3(10)^9}{48(200)(170)10^6} = 1.059 \text{ mm}$$

$$R_B = \frac{v_B}{\delta_{BB}} = \frac{8.33}{1.059} = 7.86 \text{ kN} \uparrow$$

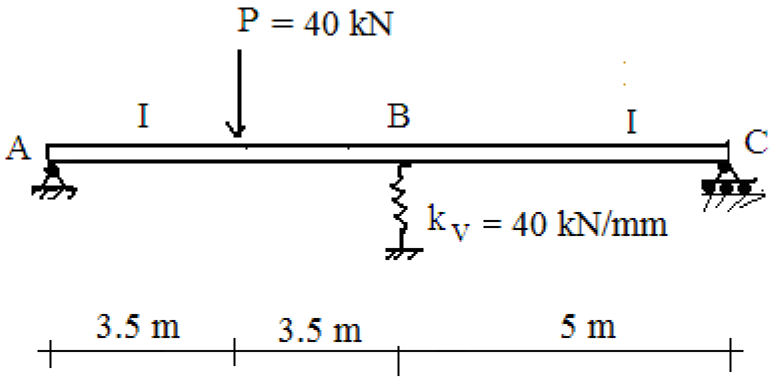
(c)

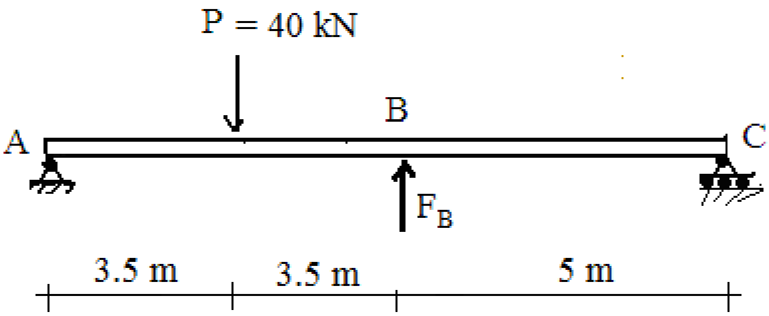


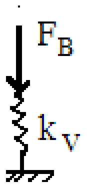
$$\delta_{BB} = \frac{L^3}{48EI} = \frac{(12)^3(10)^9}{48(200)(170)10^6} = 1.059 \text{ mm}$$

$$R_B = \frac{v_B}{\delta_{BB}} = \frac{-20}{1.059} = -18.89 \text{ kN} \qquad R_B = 18.89 \text{ kN} \downarrow$$

(d)



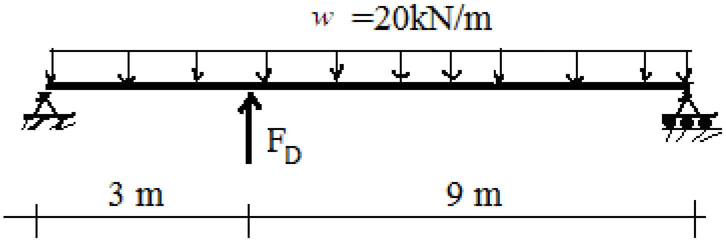


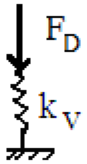


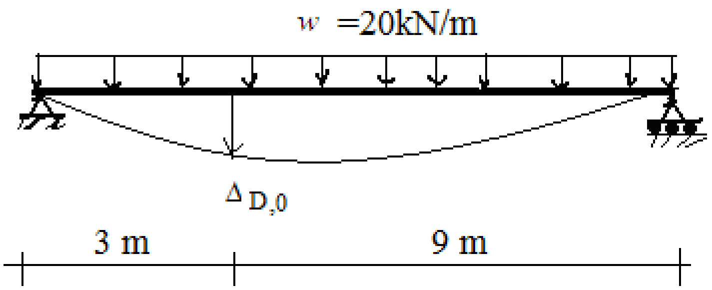
$$\left\{\begin{aligned} \Delta_{B,0} &= 25.946\frac{P}{EI} = 25.946\frac{40(10)^9}{(200)(170)(10)^6} = 30.52\ \text{mm} \\ \delta_{BB} &= \frac{L^{\ 3}}{48EI} = \frac{(12)^3(10)^9}{48(200)(170)(10)^6} = 1.059\ \text{mm} \end{aligned}\right.$$

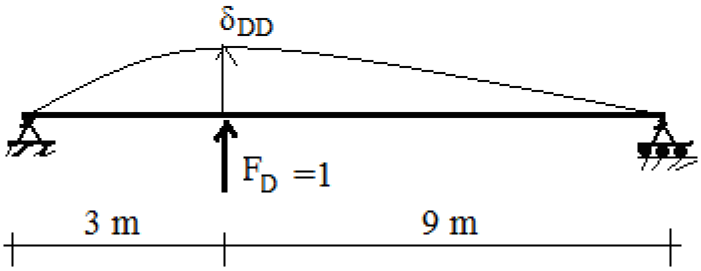
$$F_B = \frac{\Delta_{B,0}}{\delta_{BB} + \dfrac{1}{k_V}} = \frac{30.52}{1.059 + \dfrac{1}{40}} = 28.15\ \text{kN}$$

(e)







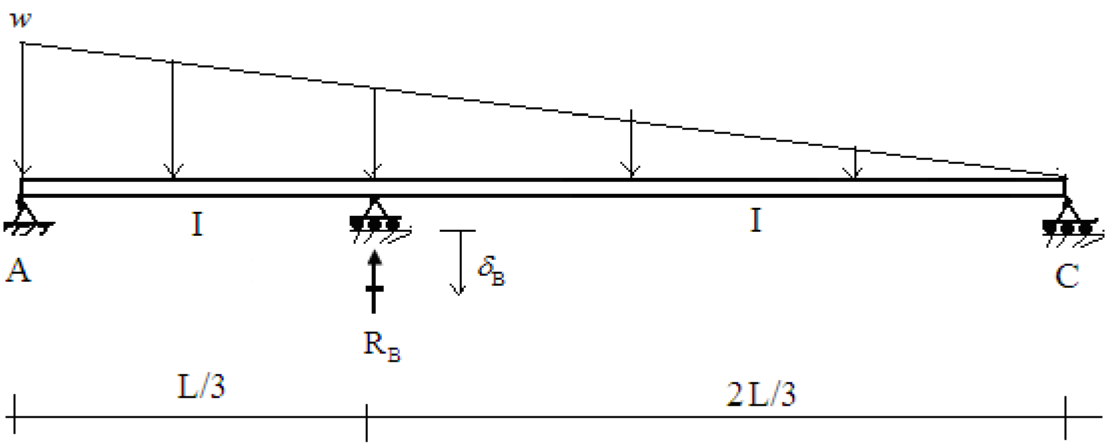


$$\begin{cases} \Delta_{D,0} = \dfrac{57wL^4}{6144EI} = \dfrac{57(20)(12)^4(10)^9}{6144(200)(170)(10)^6} = 113.16 \text{ mm} \\ \delta_{DD} = \dfrac{3L^3}{256EI} = \dfrac{3(12)^3(10)^9}{256(200)(170)(10)^6} = 0.596 \text{ mm} \end{cases}$$

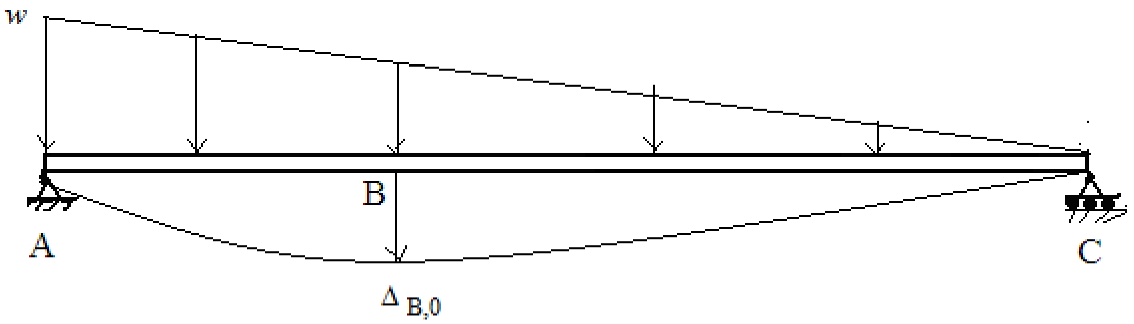
$$F_D = \frac{\Delta_{D,0}}{\delta_{DD} + \dfrac{1}{k_V}} = \frac{113.16}{0.596 + \dfrac{1}{40}} = 17.97 \text{ kN}$$

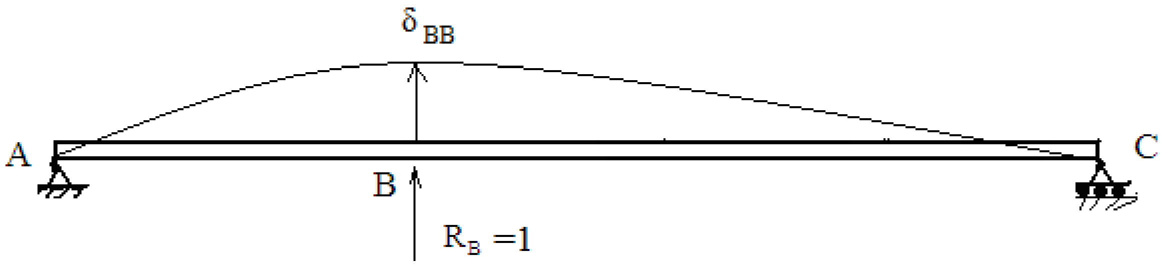
Problem 9.5

$I = 400 \text{ in}^4$, $E = 29,000$ ksi, $L = 54$ ft, $w = 2.1$ kip/ft, $\delta_B = 1.2$ in



(i) The distributed load shown





↑+ $$-\Delta_{B,0} + R_B\,\delta_{BB} = 0$$

The deflection terms are given in Chapter 3

$$\begin{cases} \Delta_{B,0} = -\dfrac{4wL^4}{729\ EI} = 14.6\ \text{in} \\ \delta_{BB} = \dfrac{4L^3}{243EI} = .386\ \text{in} \end{cases}$$

$$R_B = \frac{\Delta_{B,0}}{\delta_{BB}} = \frac{14.6}{0.386} = 37.8 \text{ kip } \uparrow$$

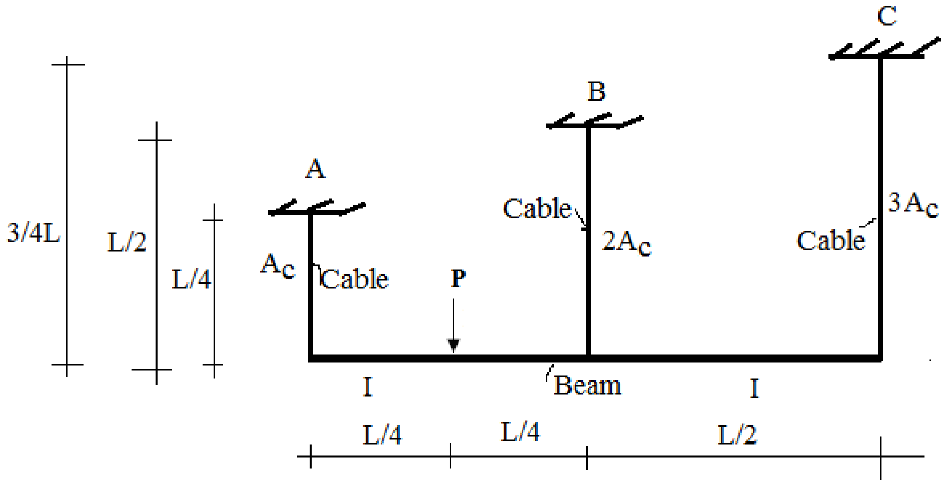
(ii) The support settlement at B

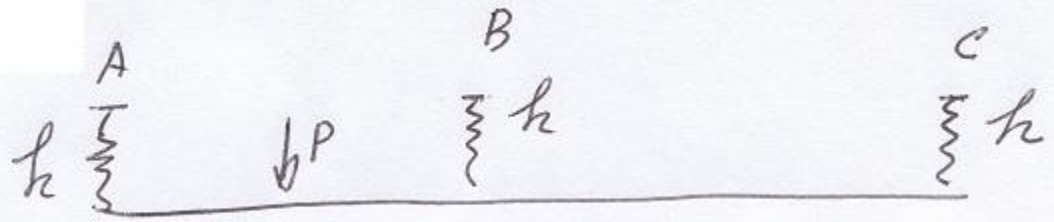
$\uparrow^{+}$ $$+R_B\,\delta_{BB} = -1.2 \text{ in}$$

$$R_B = \frac{-1.2}{\delta_{BB}} = \frac{-1.2}{0.386} = -3.1 \text{ kip} \qquad R_B = 3.1 \text{ kip } \downarrow$$

Problem 9.6

$A_C = 1200 \text{ mm}^2$, $L = 9$ m, $P = 40$ kN and $E = 200$ GPa .

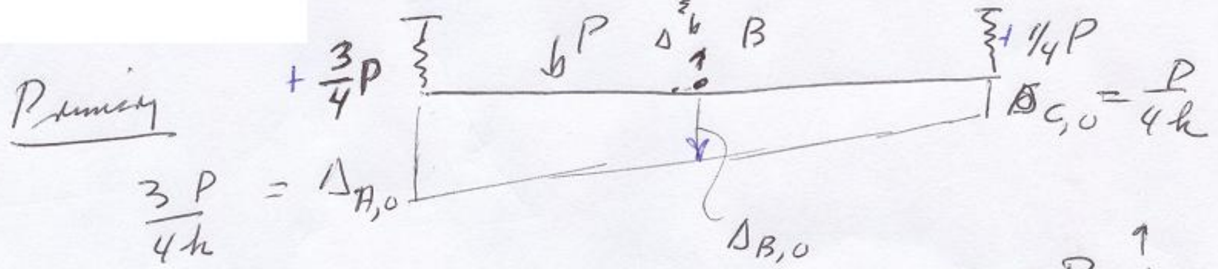




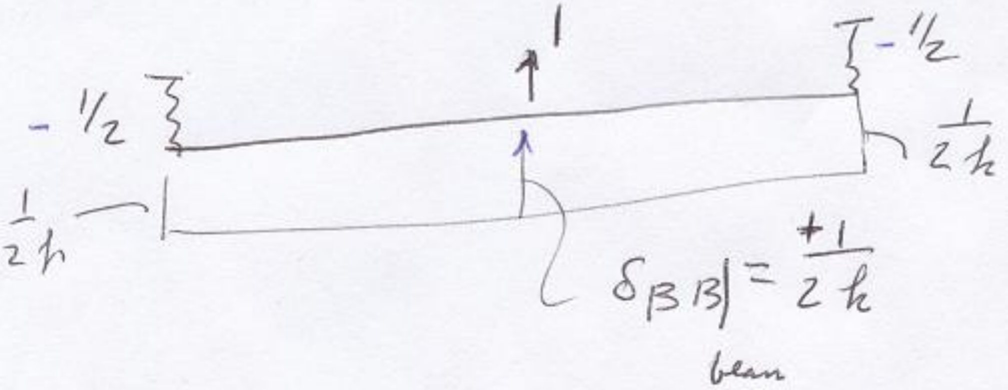
Consider cables to act as springs

$$k = \frac{AE}{L}$$

For the given properties, the spring constants are equal. Indeterminate to 1st degree. Take interior spring as redundant.

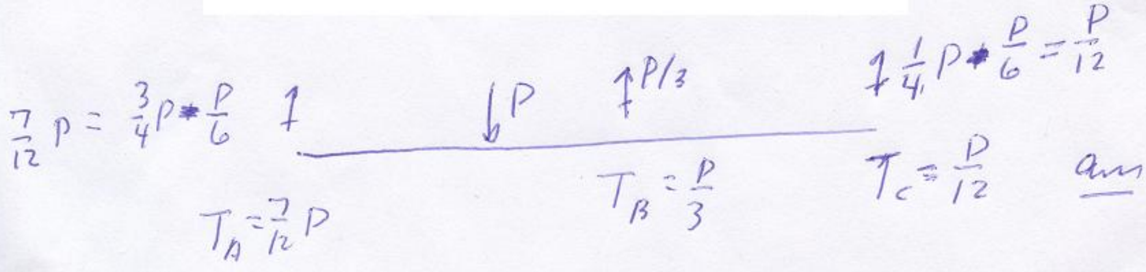


$$\Delta_{B,0} = \frac{1}{2}\left(\Delta_{A,0} + \Delta_{C,0} \right) = \frac{-P}{2k} \quad \uparrow \downarrow$$

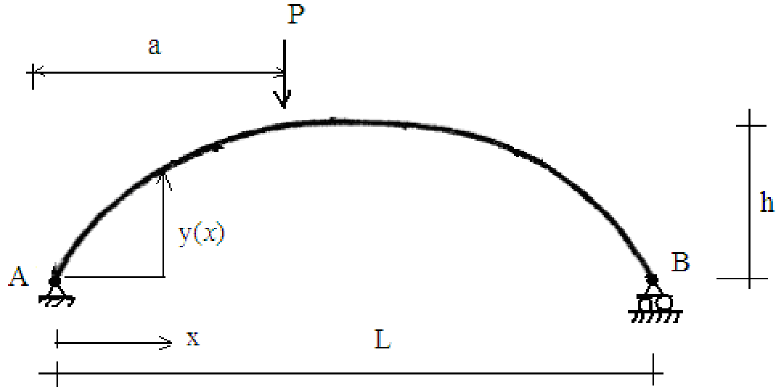


$$\delta_{BB} = \frac{1}{k} + \frac{1}{2k} = \frac{3}{2k}$$

$$\overset{\downarrow}{X_B} = -\frac{\Delta_{B,0}}{\delta_{BB}} = \frac{P/2k}{3/2k} = \frac{P}{3}$$

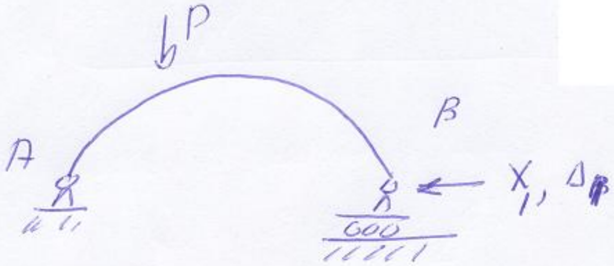


Problem 9.7



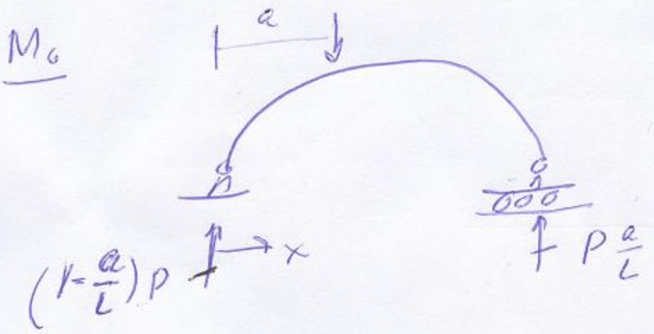
$$y = 4h\left(\frac{x}{L} - (\frac{x}{L})^2\right) \qquad I = \frac{I_o}{\cos\theta}$$

(a) Determine the horizontal reaction at B due to the concentrated load.



See Example 9.12

M_0



$M_0 = (1 - \frac{a}{L})Px \qquad 0 < x < a$

$M_0 = (L - x)P\frac{a}{L} \qquad a > x < L$

δM for $X = 1$

$\delta M = -y$

Neglect axial and shear deformation.

$$X_1 = -\frac{\Delta_{1,0}}{\delta_{11}} = -\frac{\int M_0 \, \delta M \frac{ds}{EI}}{\int \delta M^2 \frac{ds}{EI}} = -\frac{\int M_0 \, \delta M \frac{dx}{EI\cos\theta}}{\int \delta M^2 \frac{dx}{EI\cos\theta}}$$

Note $I\cos\theta = I_0$ and $\delta M = -y$.

$$X_1 = \frac{\int M_0 \, y \, dx}{\int y^2 dx}$$

Dimensionless form

$$\bar{x} = \frac{x}{L} \qquad \bar{a} = \frac{a}{L}$$

$$M_0 = (1-\bar{a})PL\bar{x} \qquad 0 < \bar{x} < \bar{a}$$

$$M_0 = PL(1-\bar{x})\bar{a} \qquad \bar{a} > \bar{x} > 1$$

$$y = \cancel{\frac{4h}{L}} \; 4h\left(\bar{x} - \frac{\bar{x}^2}{\cancel{L}}\right)$$

$$X_1 = \frac{\int_0^1 M_0(\bar{x}) \ast y(\bar{x}) \, d\bar{x}}{\int_0^1 (y(\bar{x}))^2 \, d\bar{x}} = \frac{f}{g}$$

$$g = 16h^2 \int_0^1 \left(\bar{x}^2 - \bar{x}^3 + \frac{\bar{x}^4}{4} \right) d\bar{x} = 16h^2 \left[\frac{1}{3} - \frac{1}{4} + \frac{1}{\underset{5}{\cancel{20}}} \right] = \frac{8}{15} h^2$$

$$f_1 = \int_0^{\bar{a}} (\quad) d\bar{x} \qquad f_2 = \int_{\bar{a}}^1 (\quad) d\bar{x}$$

$$f_1 = PL(1-\bar{a}) \int_0^{\bar{a}} \bar{x} \left[4h(\bar{x} - \bar{x}^2) \right] d\bar{x}$$

$$= 4hPL(1-\bar{a}) \int (\bar{x}^2 - \bar{x}^3) d\bar{x} = 4hPL(1-\bar{a}) \left[\frac{\bar{a}^3}{3} - \frac{\bar{a}^4}{4} \right]$$

$$= \frac{4PLh}{3} (1-\bar{a}) \bar{a}^3 \left(1 - \frac{3}{4}\bar{a}\right) = \frac{4PLh}{3} \left[\bar{a}^3 \left(1 - \frac{7}{4}\bar{a} + \frac{3}{4}\bar{a}^2\right) \right]$$

$$f_2 = PL\bar{a} \int_{\bar{a}}^1 (1-\bar{x}) \left[4h(\bar{x} - \bar{x}^2) \right] d\bar{x}$$

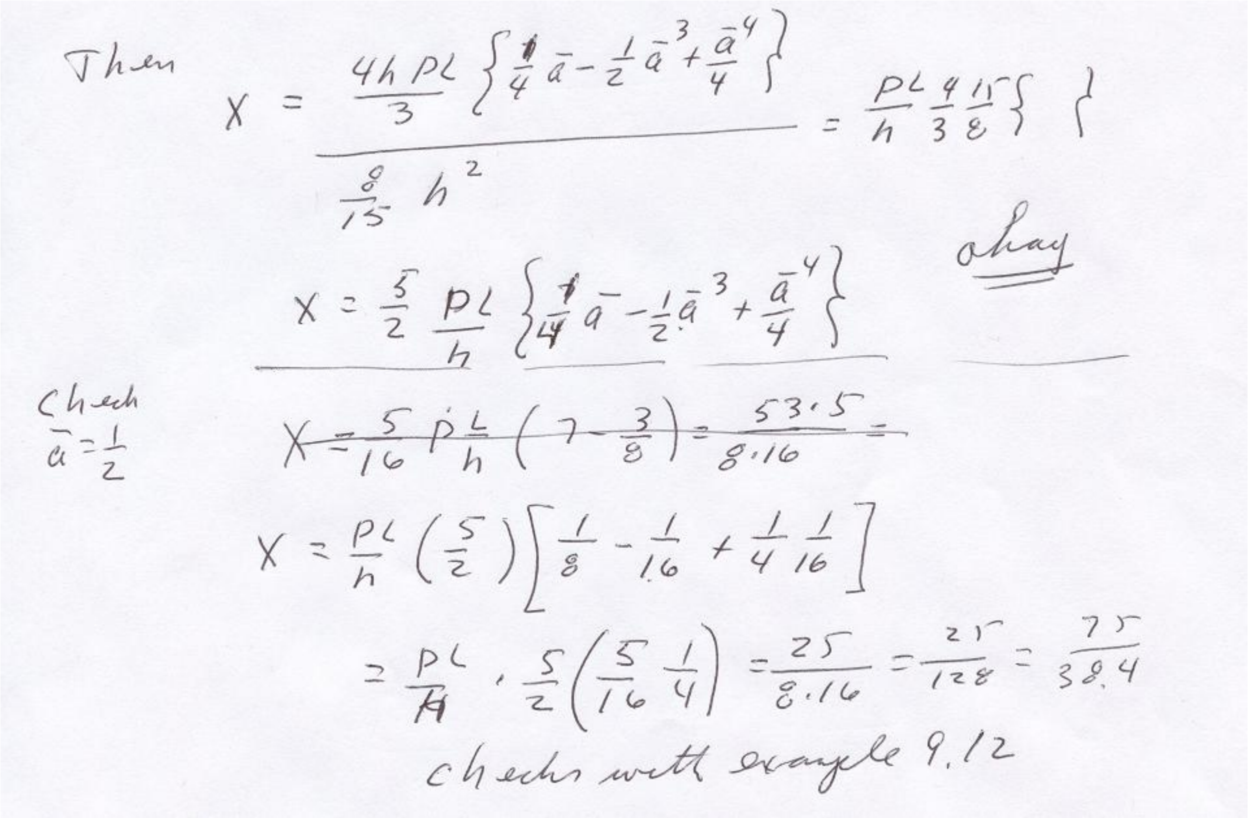
$$= 4hPL\bar{a} \int_{\bar{a}}^1 \left\{ \bar{x} - \bar{x}^2 - \bar{x}^2 + \bar{x}^3 \right\} d\bar{x} = 4hPL\bar{a} \left[\frac{\bar{x}^2}{2} - \frac{2\bar{x}^3}{3} + \frac{\bar{x}^4}{4} \right]_{\bar{a}}^1$$

$$f_2 = 4hPL\bar{a} \left\{ \frac{1}{2} - \frac{2}{3} + \frac{1}{4} - \left[\frac{\bar{a}^2}{2} - \frac{2\bar{a}^3}{3} + \frac{\bar{a}^4}{4} \right] \right\}$$

$$f_2 = \frac{4hPL\bar{a}}{\cancel{3}} \left\{ \frac{1}{\underset{4}{\cancel{12}}} - \frac{3\bar{a}^2}{2} \left[1 - \frac{4}{3}\bar{a} + \frac{1}{2}\bar{a}^2 \right] \right\}$$

$$f = \frac{4hPL}{3} \left\{ \bar{a}^3 - \frac{7}{4}\bar{a}^4 + \frac{3}{4}\bar{a}^5 + \frac{1}{4}\bar{a} - \frac{3}{2}\bar{a}^3 \left(1 - \frac{4}{3}\bar{a} + \frac{1}{2}\bar{a}^2\right) \right\}$$

$$f = \frac{4hPL}{3} \left\{ -\frac{1}{2}\bar{a}^3 + \frac{1}{4}\bar{a} + \frac{\bar{a}^4}{4} \right\}$$



Then

$$X = \frac{\frac{4hPL}{3}\left\{\frac{1}{4}\bar{a} - \frac{1}{2}\bar{a}^3 + \frac{\bar{a}^4}{4}\right\}}{\frac{8}{15}h^2} = \frac{PL}{h}\frac{4}{3}\frac{15}{8}\left\{\right\}\left\{\right.$$

$$X = \frac{5}{2}\frac{PL}{h}\left\{\frac{1}{4}\bar{a} - \frac{1}{2}\bar{a}^3 + \frac{\bar{a}^4}{4}\right\}$$

okay

Check $\bar{a} = \frac{1}{2}$

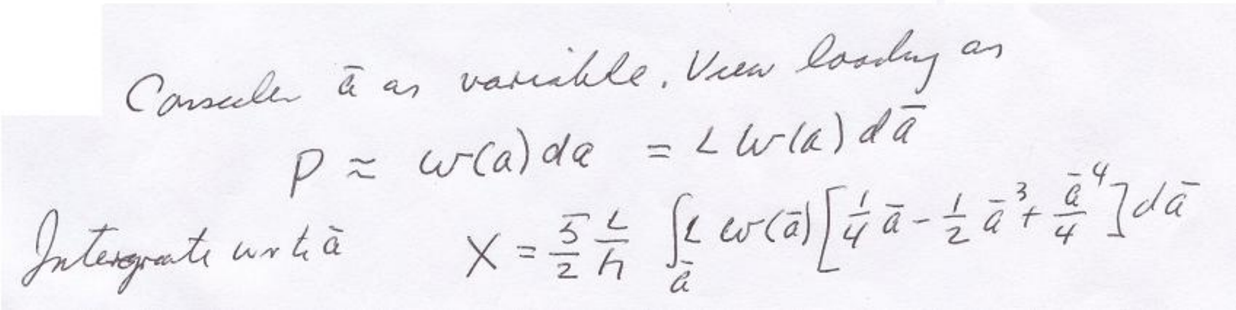
~~$X = \frac{5}{16}\frac{PL}{h}\left(7 - \frac{3}{8}\right) = \frac{53.5}{8\cdot 16} =$~~

$$X = \frac{PL}{h}\left(\frac{5}{2}\right)\left[\frac{1}{8} - \frac{1}{16} + \frac{1}{4}\frac{1}{16}\right]$$

$$= \frac{PL}{h}\cdot\frac{5}{2}\left(\frac{5}{16}\frac{1}{4}\right) = \frac{25}{8\cdot 16} = \frac{25}{128} = \frac{75}{384}$$

checks with example 9.12

(b) Utilize the results of part (a) to obtain an analytical expression for the horizontal reaction due to a distributed loading, $w(x)$.



Consider $\bar{a}$ as variable. View loading as

$$P \approx w(a)\,da = L\,w(a)\,d\bar{a}$$

Integrate wrt $\bar{a}$ $\quad X = \frac{5}{2}\frac{L}{h}\int_{\bar{a}}^{L} w(\bar{a})\left[\frac{1}{4}\bar{a} - \frac{1}{2}\bar{a}^3 + \frac{\bar{a}^4}{4}\right]d\bar{a}$

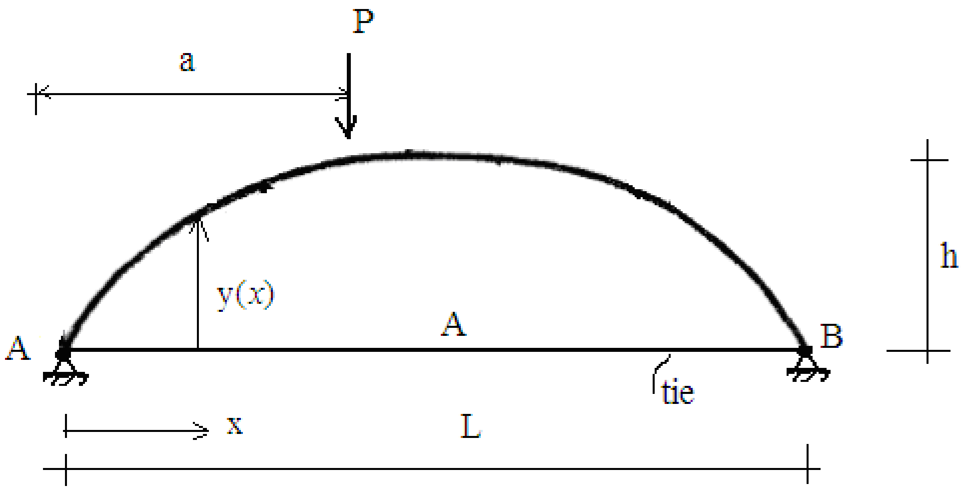
(c) Specialize (b) for a uniform loading, $w(x) = w_0$.

Assume $w(a) = w_0$ = const.

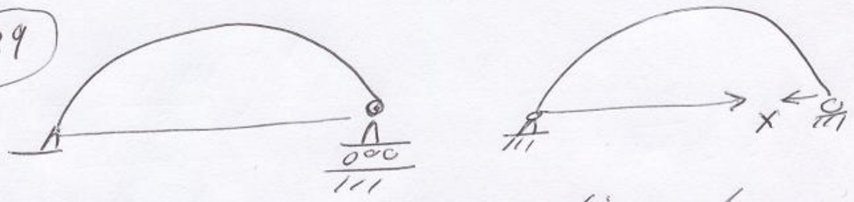
$$X = \frac{5}{2}\frac{L^2}{h} w_0 \int_0^1 \left(\frac{\bar{a}}{4} - \frac{\bar{a}^3}{2} + \frac{\bar{a}^4}{4}\right) d\bar{a}$$

$$= \frac{wL^2}{h}\left(\frac{5}{2}\right)\left[\frac{\bar{a}^2}{8} - \frac{\bar{a}^4}{8} + \frac{\bar{a}^5}{5\cdot 4}\right] = \frac{w_0 L^2}{8h} \quad \underline{\text{ans}}$$

(d) Suppose the horizontal support at B is replaced by a member extending from A to B. Repeat part (a).



Note Eq. 9.39



Modify δ_{11} to include extension of tie member.

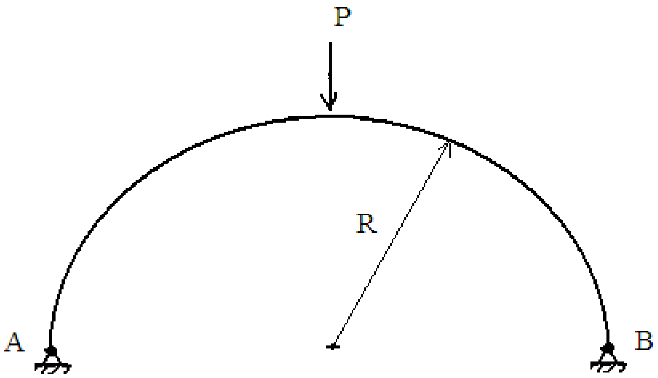
$$\delta_{11} = \int_0^L \frac{\delta m^2 dx}{E I_0} + \frac{L}{A_t E_t}$$

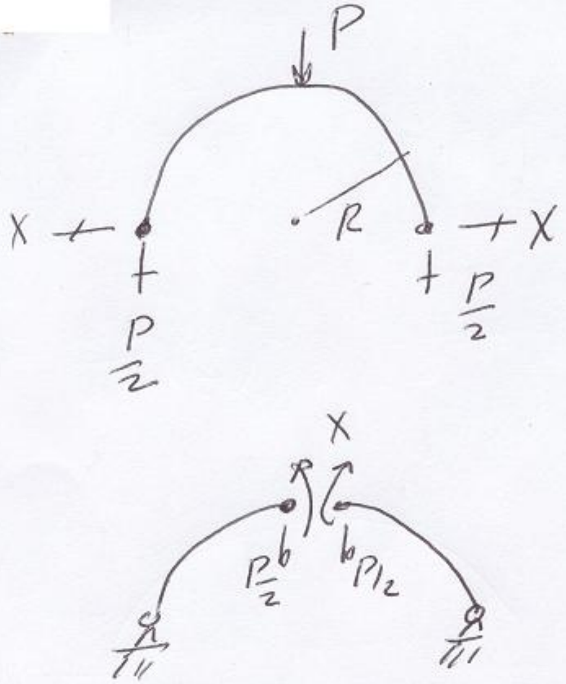
$$= \frac{1}{E I_0} \left\{ \int_0^L y^2 dx + \frac{I L}{A} \right\}$$

$$= \frac{1}{E I_0} \left\{ \frac{8}{15} h^2 + \frac{I_0 L}{A} \right\}$$

Problem 9.8

Consider the semi-circular arch shown below. Determine the distribution of the axial and shear forces and the bending moment. The cross section properties are constant.

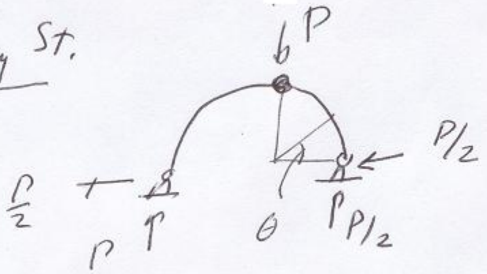




Take Moment at center as the redundant

Primary Str is 3 hinged arch

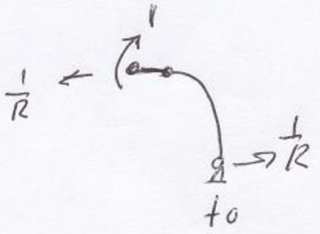
Primary St.



$$M_0(\theta) = -\frac{P}{2} R\sin\theta + \frac{P}{2} R(1-\cos\theta)$$

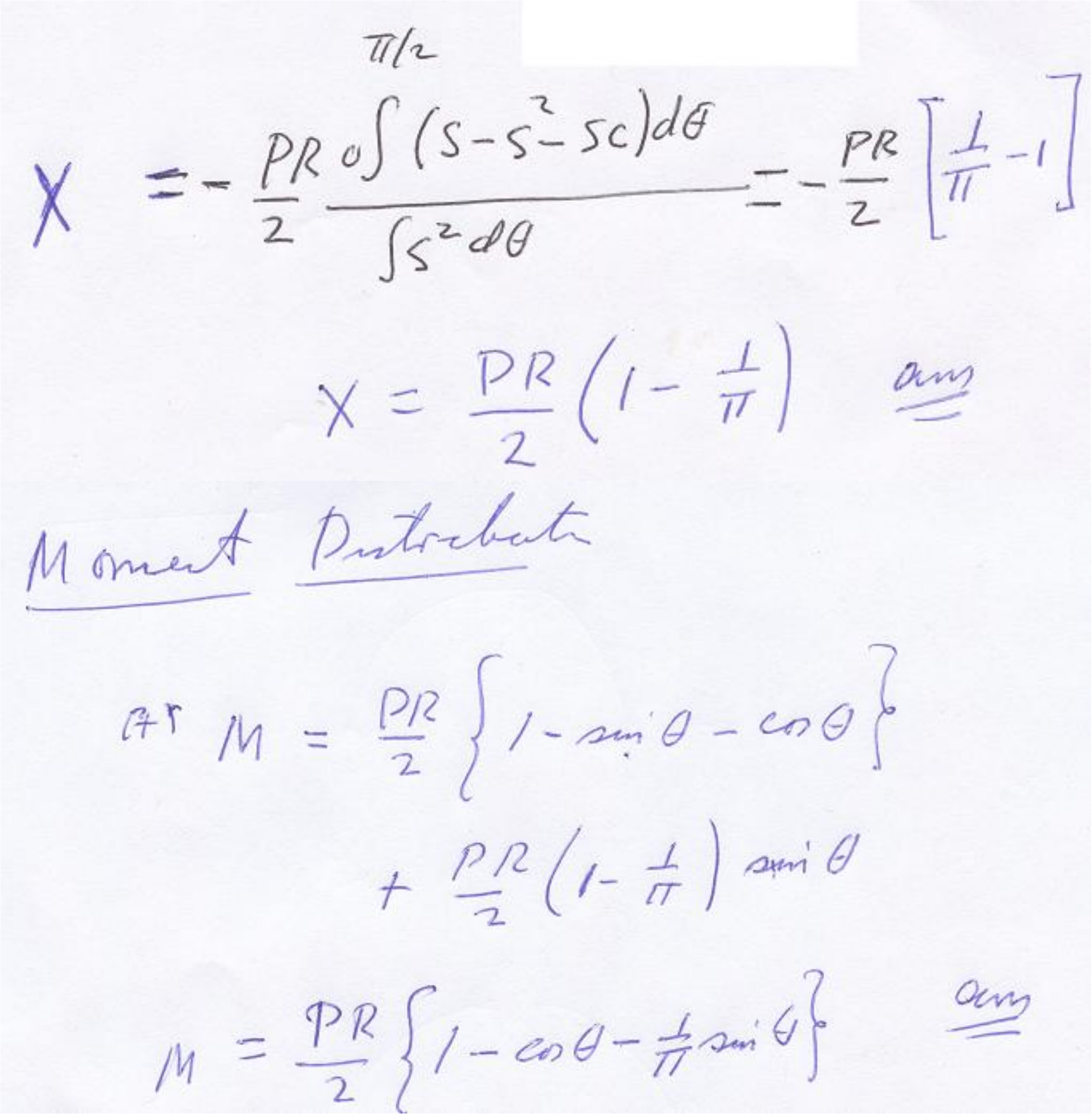
$$M_0 = \frac{PR}{2}\left\{ 1-\sin\theta-\cos\theta \right\}$$

X=1



$$\delta M = \frac{1}{R}\left(R\sin\theta\right) = \sin\theta$$

$$X = \frac{-\Delta_{1,0}}{\delta_{11}} = -\frac{{}_0\int \frac{PR}{2}\left\{(1-s-c) s R d\theta\right\}}{\int s^2 R d\theta}$$



$$X = -\frac{PR}{2}\frac{\int_0^{\pi/2}(s-s^2-sc)\,d\theta}{\int s^2\,d\theta} = -\frac{PR}{2}\left[\frac{1}{\pi}-1\right]$$

$$X = \frac{PR}{2}\left(1-\frac{1}{\pi}\right) \quad \underline{\underline{\text{ans}}}$$

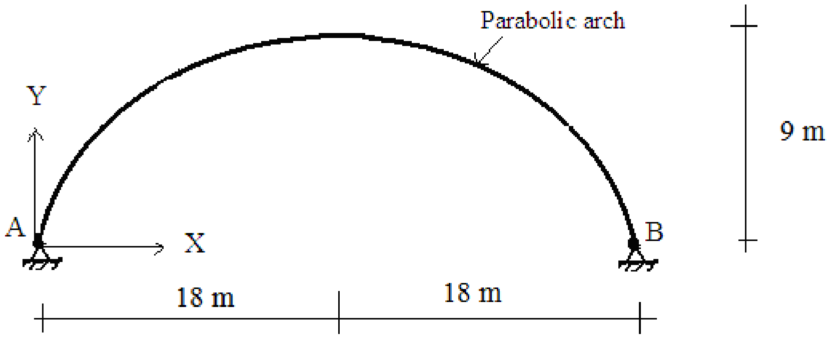
<u>Moment Distribution</u>

$$\text{At } M = \frac{PR}{2}\left\{1-\sin\theta-\cos\theta\right\} + \frac{PR}{2}\left(1-\frac{1}{\pi}\right)\sin\theta$$

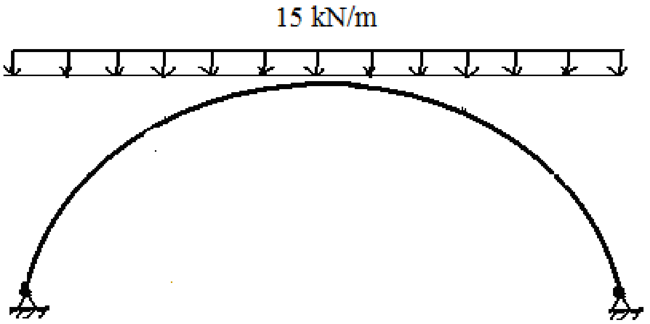
$$M = \frac{PR}{2}\left\{1-\cos\theta-\frac{1}{\pi}\sin\theta\right\} \quad \underline{\underline{\text{ans}}}$$

Problem 9.9

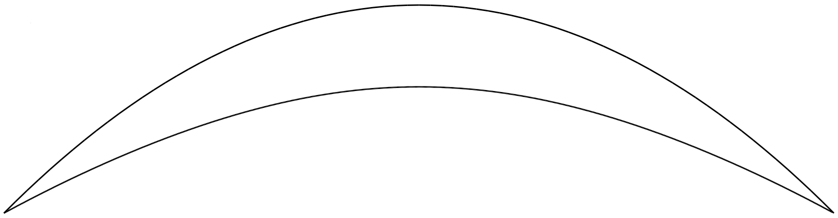
Take A = 20,000 mm^2 I = $400(10)^6$ mm^4 and E = 200 GPa



Case (a):

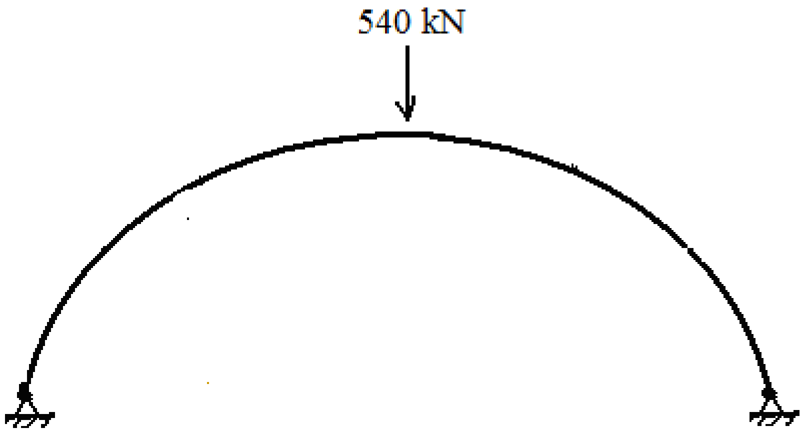


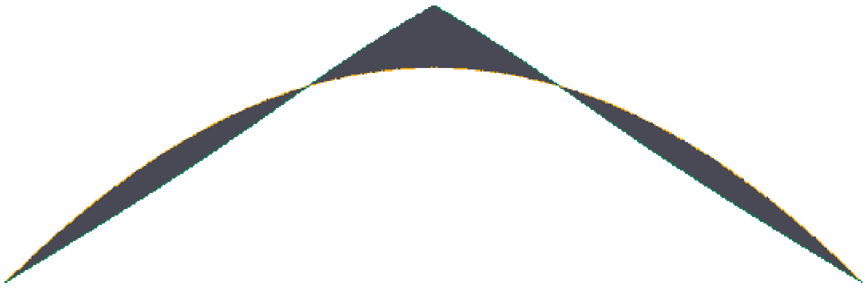
$M_{max} = 0$ (no bending moment)



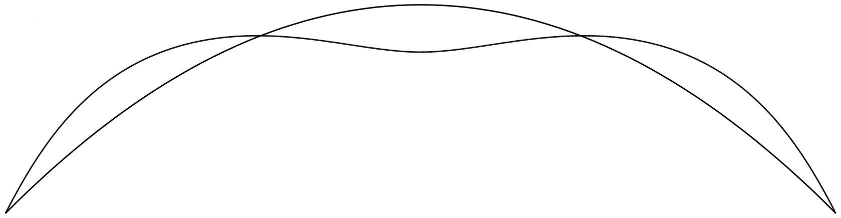
Deflected shape

Case (b):



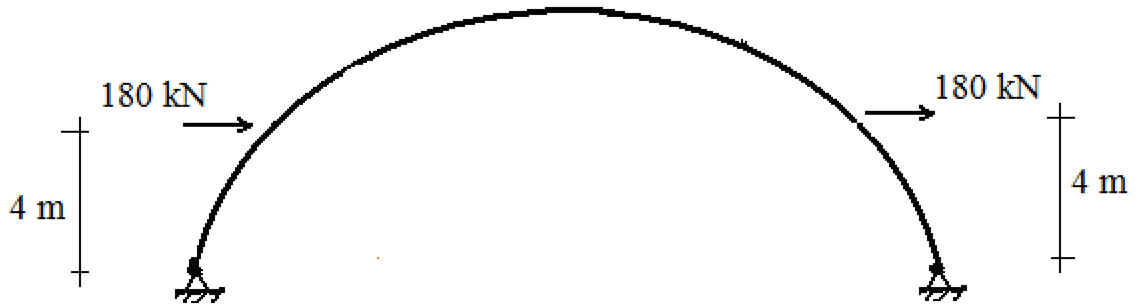


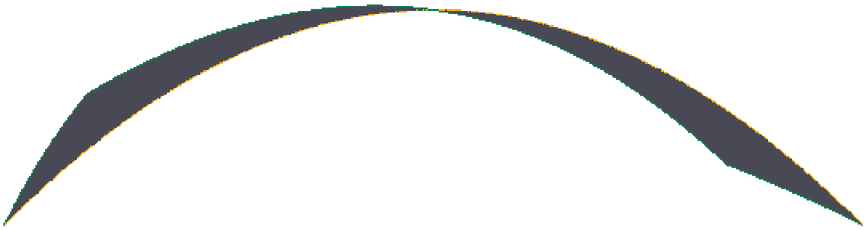
M_{max} =1101 kN-m



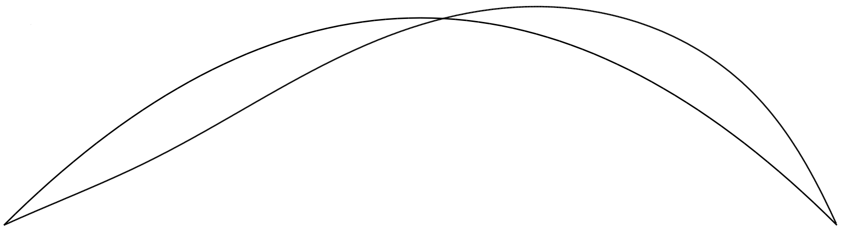
Deflected shape

Case (c):





$M_{max} = 529$ kN-m

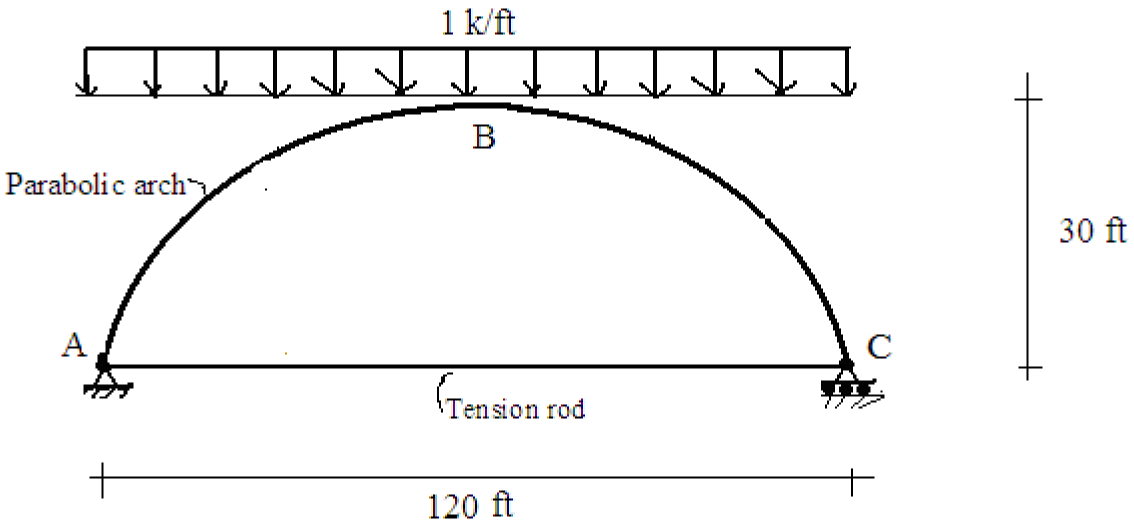


Deflected shape

Problem 9.10

Consider the following values for the area of the tension rod AC: $4 \text{ in}^2, 8 \text{ in}^2, 16 \text{ in}^2$.

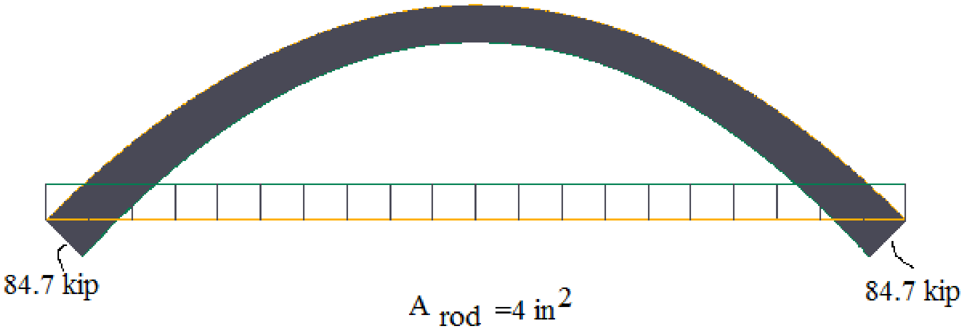
$A = 30 \text{ in}^2 \quad I = 1000 \text{ in}^4 \quad E = 29{,}000$ ksi .



$A = 30 \text{ in}^2 \quad I = 1000 \text{ in}^4 \quad E = 29{,}000$ ksi

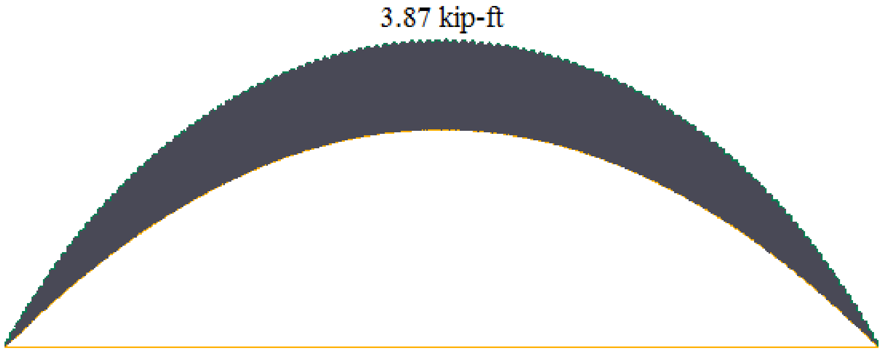
$A_{rod} = 4 \text{ in}^2$

Bending moment



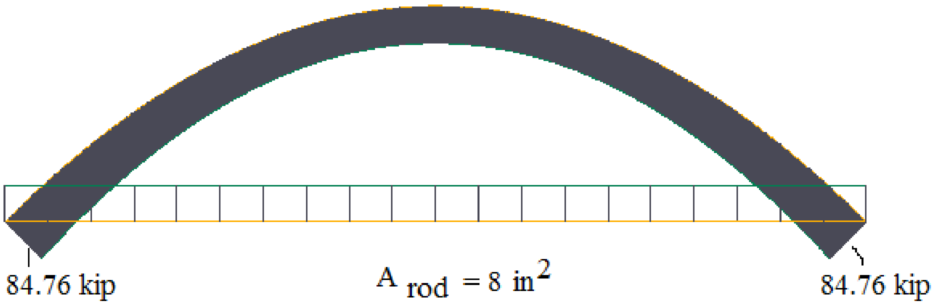
$A_{rod} = 4 \text{ in}^2$

Axial force

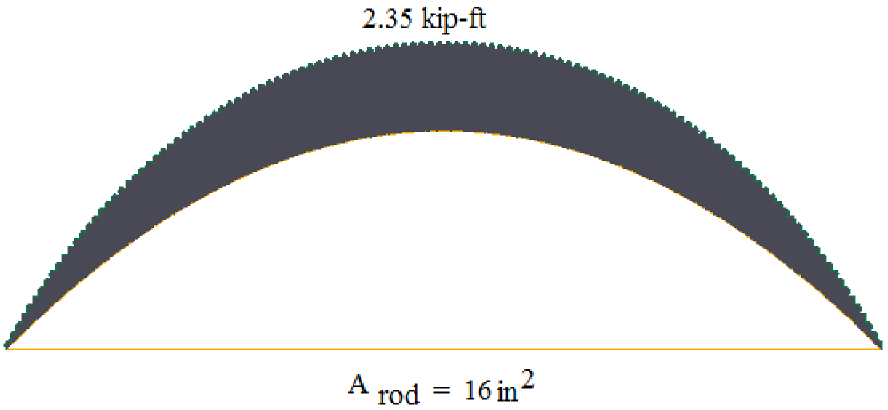


$A_{rod} = 8 \text{ in}^2$

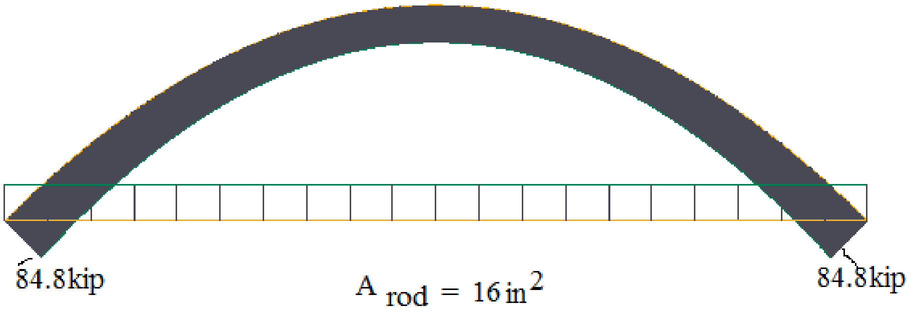
Bending moment



Axial force

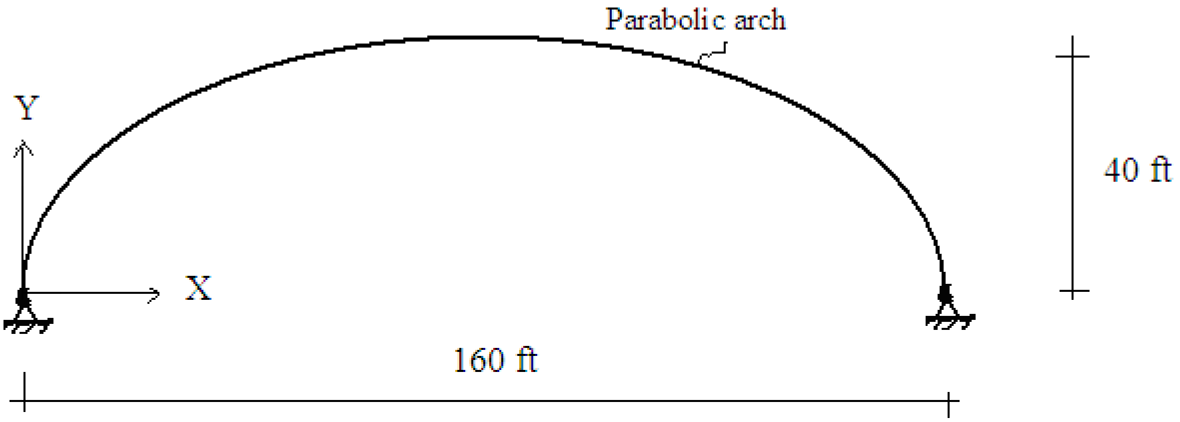


Bending moment



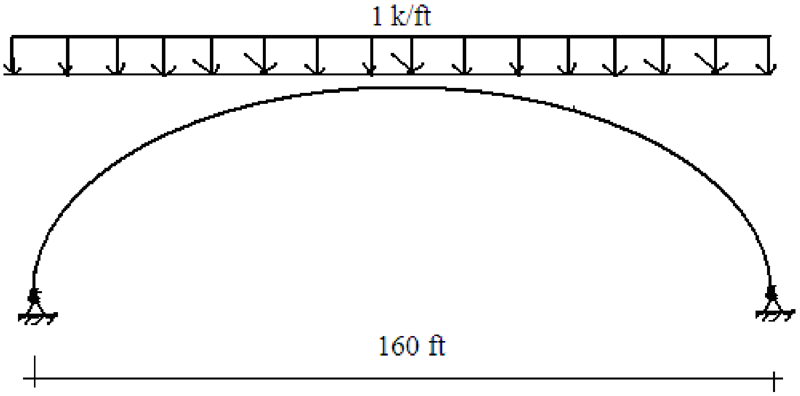
Axial force

Problem 9.11



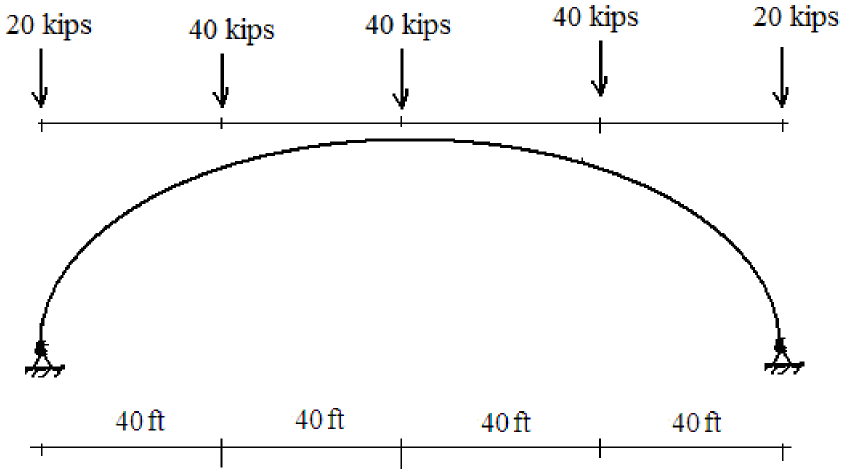
$A = 40 \text{ in}^2 \quad I = 1200 \text{ in}^4 \quad E = 29{,}000 \text{ ksi}$

Case(a):



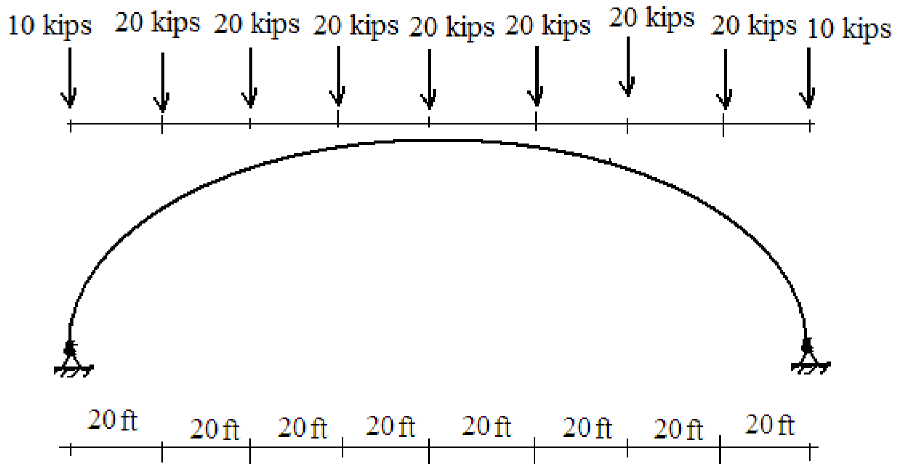
$M_{max} = 0$ (no bending moment)

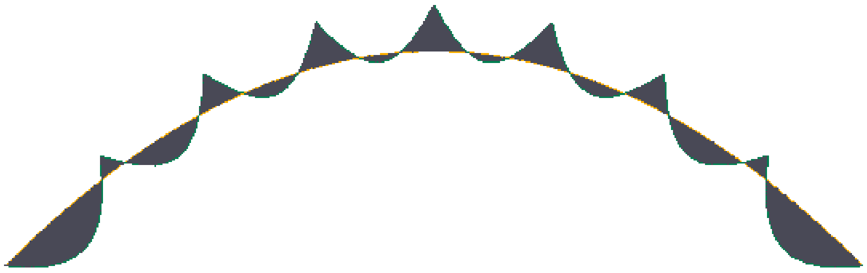
Case (b):



$M_{max} = 130$ kip-ft

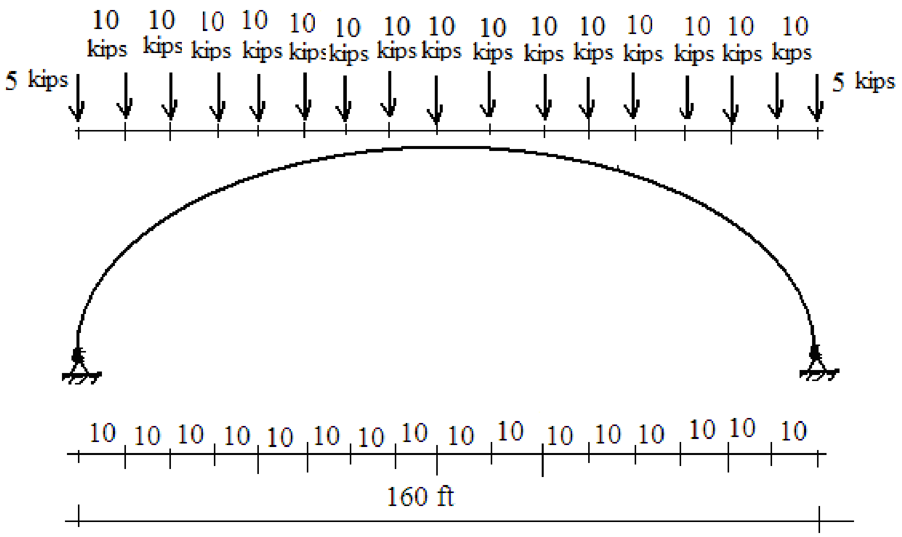
Case (c):

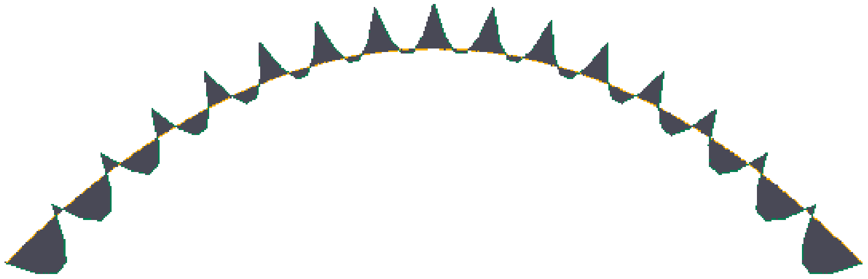




$M_{max} = 43.3$ kip-ft

Case (d):

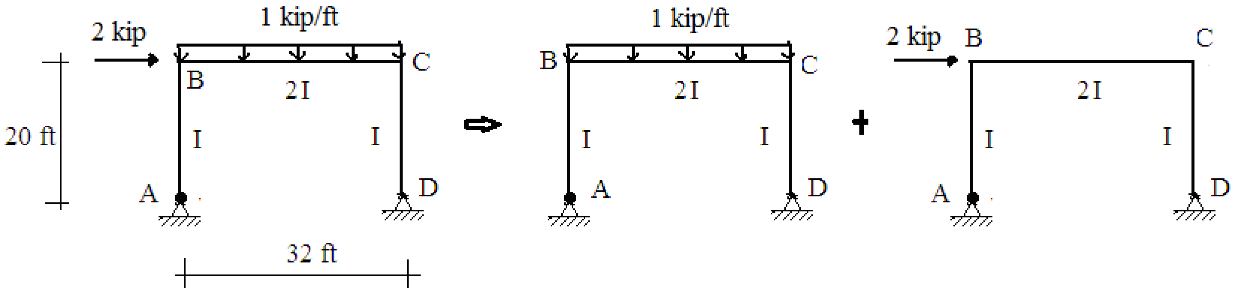




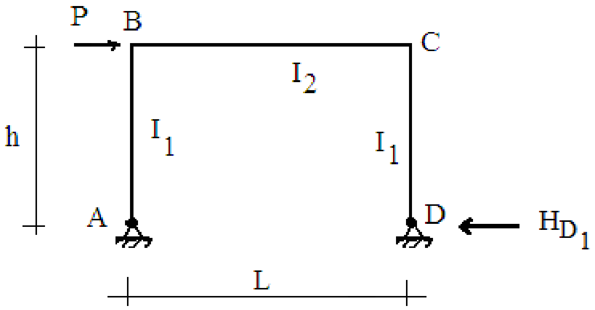
$M_{max} = 10.9$ kip-ft

Problem 9.12

Determine the horizontal reaction at support D.

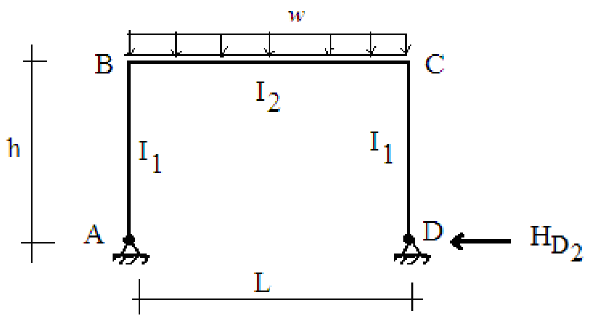


(See Figure 9.38) $H_{D_1} = \frac{P}{2} = 1 \text{ kip} \leftarrow$

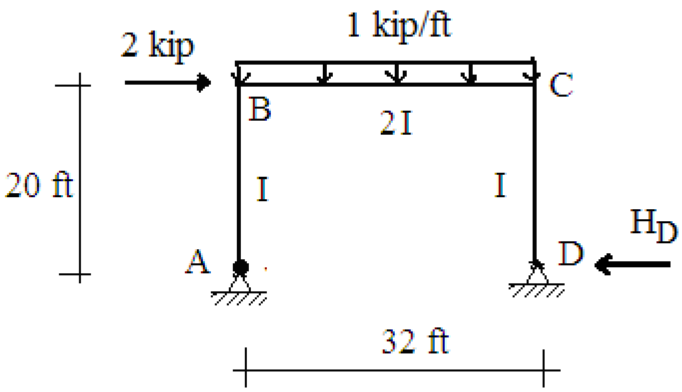


(See example 9.15) $r_g = \frac{I_2}{L} = \frac{I}{16}$ $r_c = \frac{I_1}{h} = \frac{I}{20}$ $\frac{r_g}{r_c} = \frac{5}{4}$

$$H_{D_2} = \frac{wL^2}{12h} \frac{1}{1 + \frac{2}{3}\frac{r_g}{r_c}} = \frac{(32)^2}{12(20)} \frac{1}{1 + \frac{2}{3}(\frac{5}{4})} = 2.33 \text{ kip} \leftarrow$$

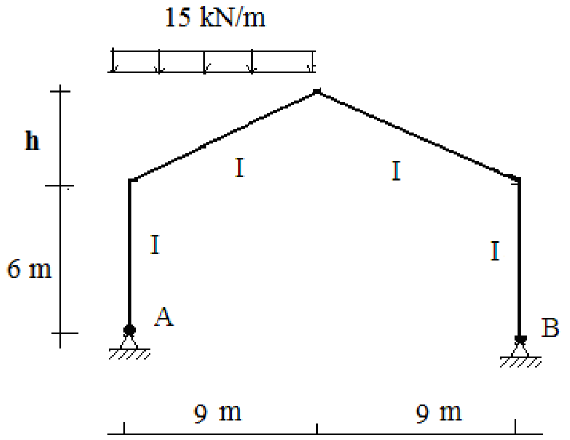


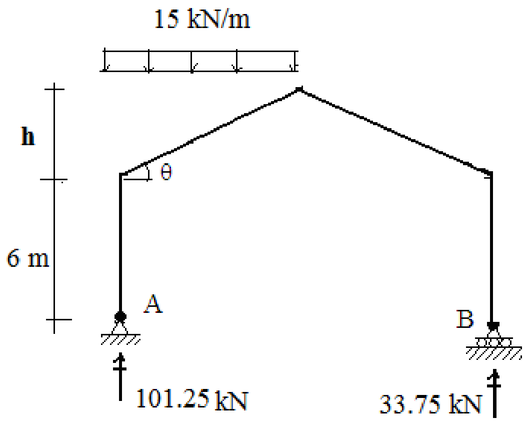
$$\therefore \quad H_D = H_{D_1} + H_{D_2} = -1 + 2.33 = 1.33 \text{ kip} \leftarrow$$

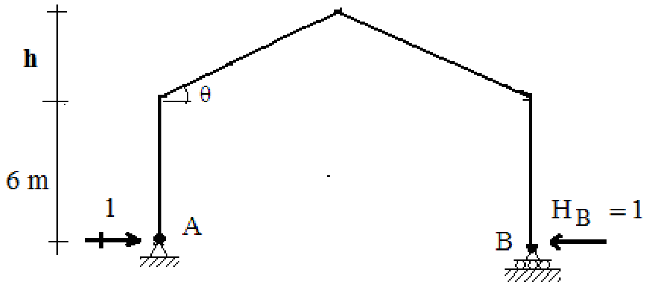


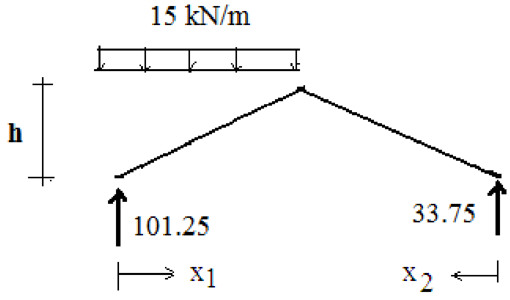
Problem 9.13

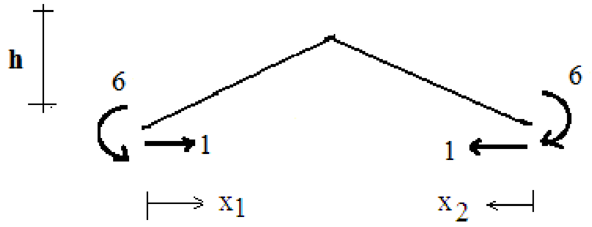
Consider h = 2 m, 4 m, 6 m.











$0 \le x_1 \le 9 \qquad M_0(x_1) = 101.25\ x_1 - 7.5\ x_1^2 \qquad \delta M(x_1) = \frac{x_1}{9h} + 6$

$0 \le x_2 \le 9 \qquad M_0(x_2) = 101.25\ x_2 \qquad \delta M(x_2) = \frac{x_2}{9h} + 6$

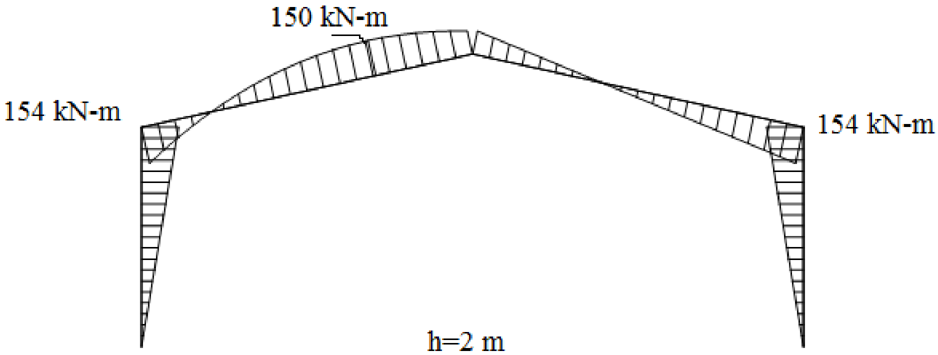
$$\Delta_{B,0}=\frac{1}{EI}\int_s M_0\ \delta M\ dS=\frac{1}{EI}\int_0^9(101.25\ x_1-7.5\ x_1{}^2)(\frac{x_1}{9h}+6)\ dx_1+\frac{1}{EI}\int_0^9(101.25\ x_2)(\frac{x_2}{9h}+6)\ dx_2$$

$$\delta_{BB}=\frac{1}{EI}\int_s(\delta M)^2\ dS=\frac{1}{EI}\int_0^9(\frac{x_1}{9h}+6)^2\ dx_1+\frac{1}{EI}\int_0^9(\frac{x_1}{9h}+6)^2\ dx_2$$

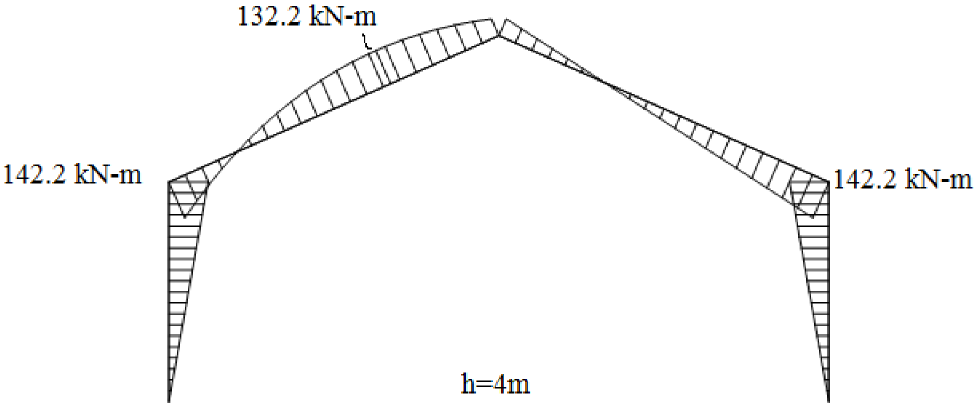
Then $H_B = -\dfrac{\Delta_{B,0}}{\delta_{BB}}$

$$EI\Delta_{B,0}=\begin{cases}0.0676(10)^6 & \text{for } h=2m\\ 0.0847(10)^6 & \text{for } h=4m\\ 0.1067(10)^6 & \text{for } h=6m\end{cases}\qquad EI\delta_{BB}=\begin{cases}-0.00263(10)^6 & \text{for } h=2m\\ -0.00357(10)^6 & \text{for } h=4m\\ -0.0049(10)^6 & \text{for } h=6m\end{cases}$$

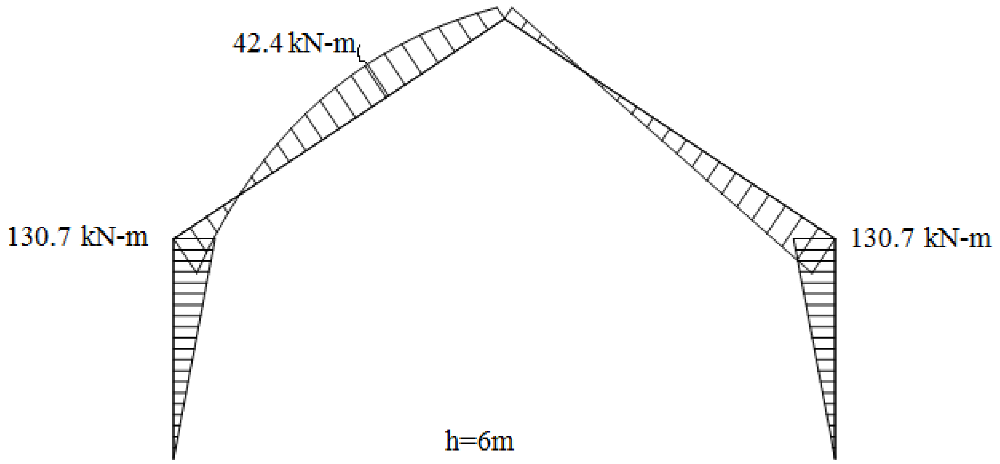
$$\therefore\qquad H_B=\begin{cases}25.7\ kN\quad\leftarrow & \text{for } h=2m\\ 23.7\ kN\quad\leftarrow & \text{for } h=4m\\ 21.78\ kN\quad\leftarrow & \text{for } h=6m\end{cases}$$



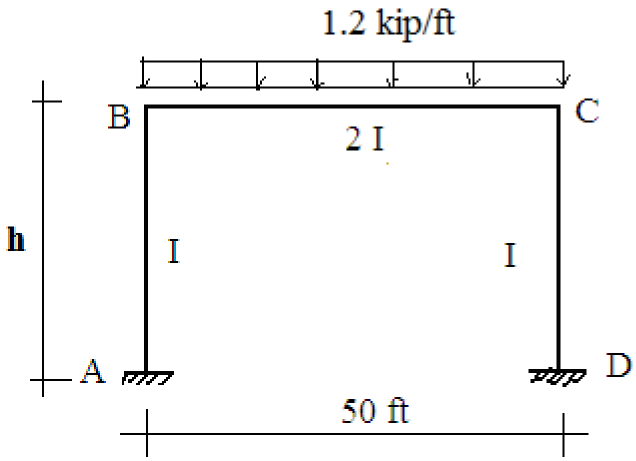
h=2 m



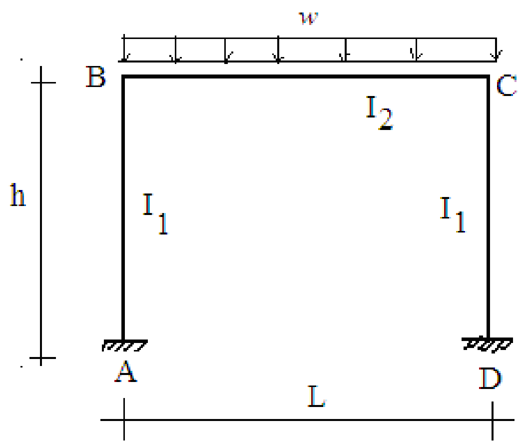
h=4m

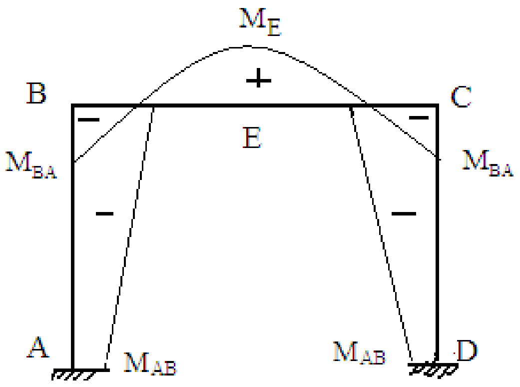


Problem 9.14
Determine the peak positive and negative moments as a function of h. Consider h = 10 ft, 20 ft, 30 ft.



See equation (9.51) and Figure 9.49





$$M_{BA} = -\frac{wL^2}{12}\frac{1}{1+\frac{1}{2}\frac{r_g}{r_c}} = -\frac{1.2(50)^2}{12}\frac{1}{1+\frac{1}{2}(\frac{h}{25})} = -\frac{250}{1+\frac{h}{50}}$$

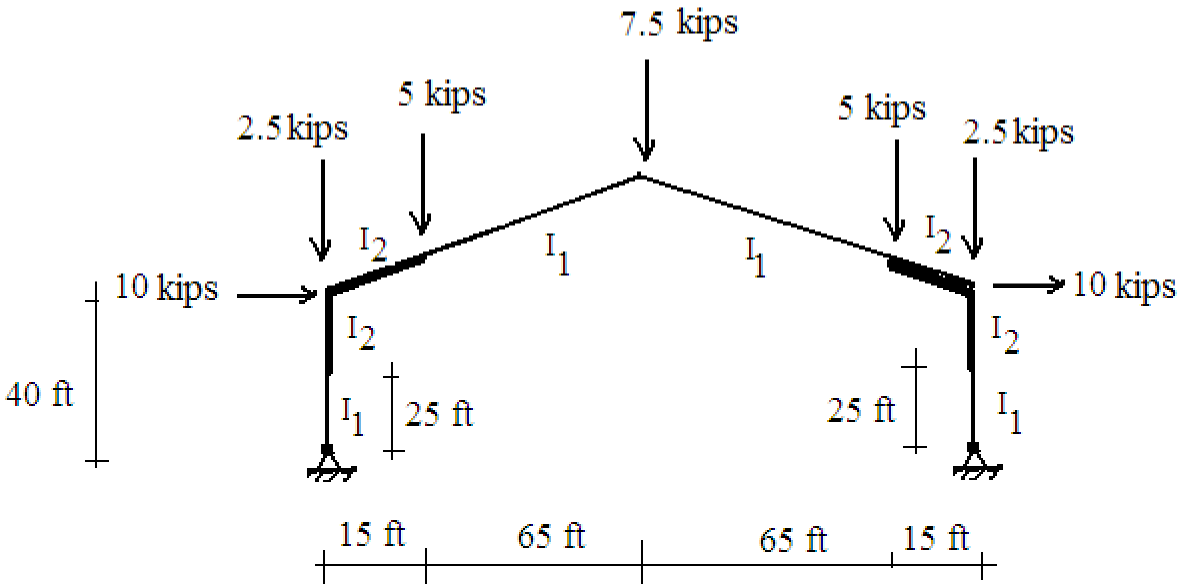
$$M_E = \frac{wL^2}{8}\left[1-\frac{2}{3}\frac{1}{1+\frac{1}{2}\frac{r_g}{r_c}}\right] = 375\left[1-\frac{2}{3(1+\frac{h}{50})}\right]$$

$$M^{-}_{max} = M_{BA} = \begin{cases} 208\ \text{kip-ft} & \text{for } h = 10\text{ft} \\ 178.6\ \text{kip-ft} & \text{for } h = 20\text{ft} \\ 156.2\ \text{kip-ft} & \text{for } h = 30\text{ft} \end{cases}$$

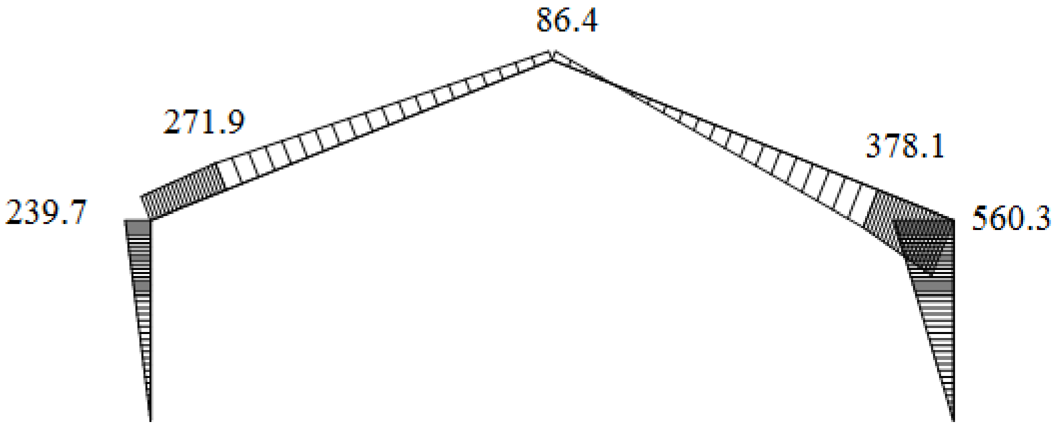
$$M^{+}_{max} = M_E = \begin{cases} 166.6\ \text{kip-ft} & \text{for } h = 10\text{ft} \\ 196.4\ \text{kip-ft} & \text{for } h = 20\text{ft} \\ 218.7\ \text{kip-ft} & \text{for } h = 30\text{ft} \end{cases}$$

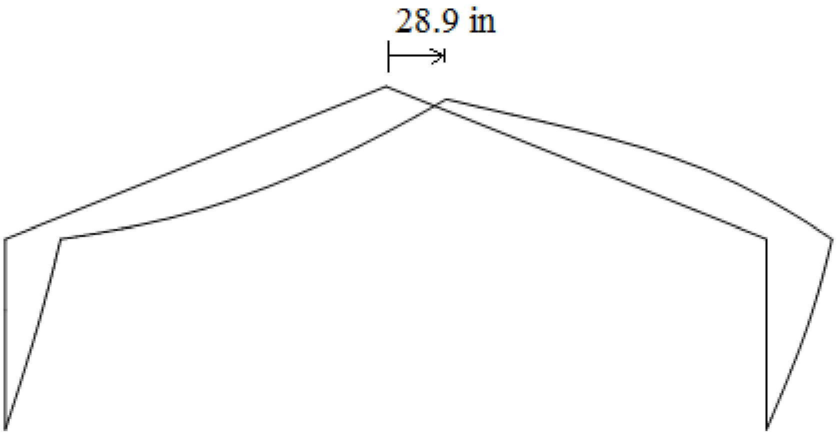
Problem 9.15

Using a Computer software system, determine the bending moment distribution and deflected shape due to the loading shown.



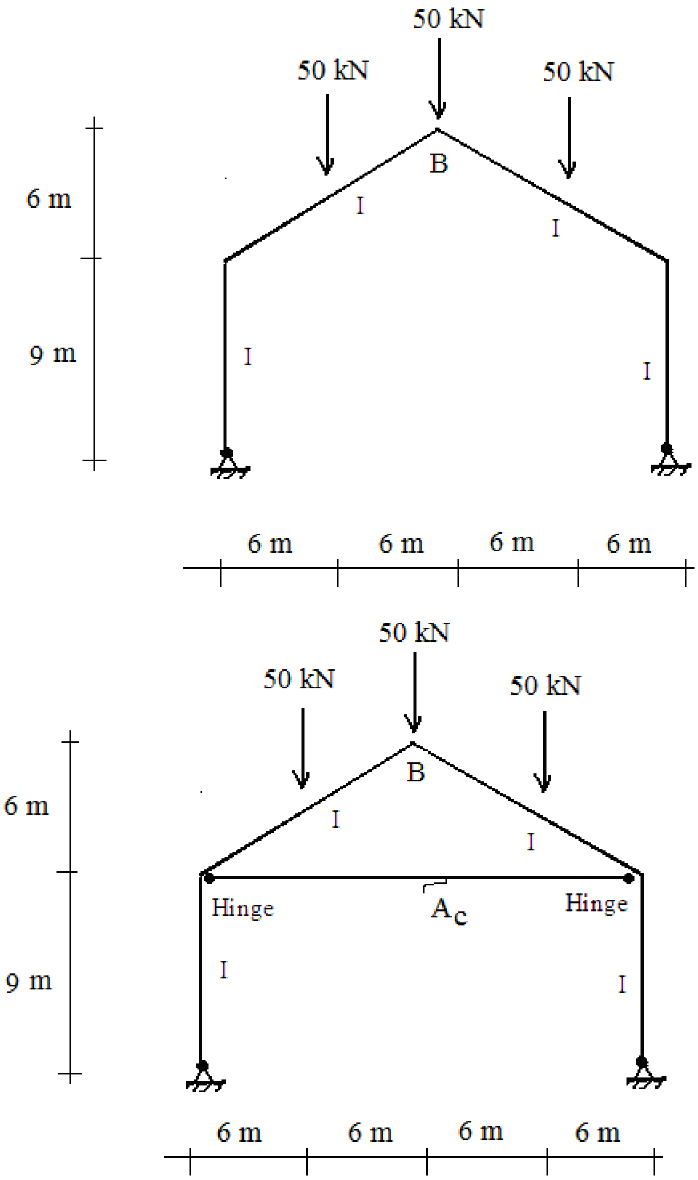
Take $I_1 = 1000\ \text{in}^4$, $I_2 = 2000\ \text{in}^4$, E=29,000 ksi and A=20 in^2 all members



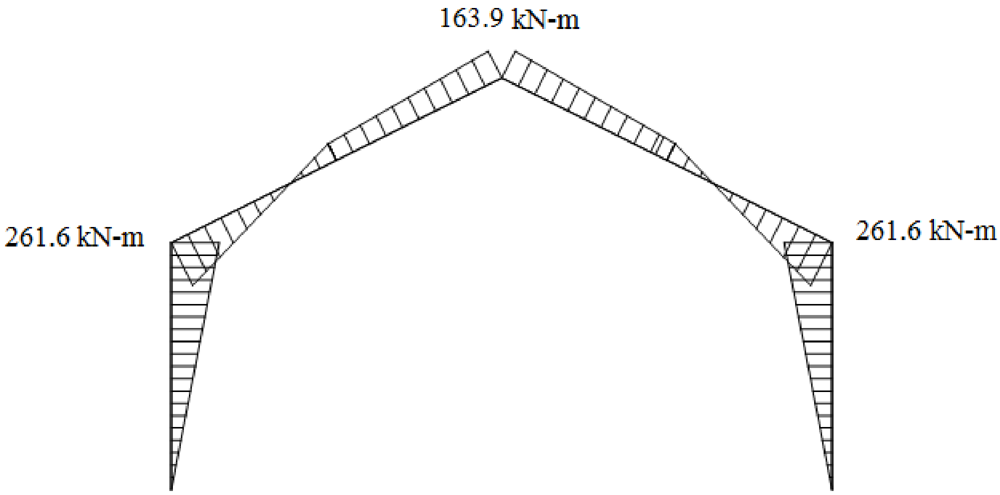


Problem 9.16

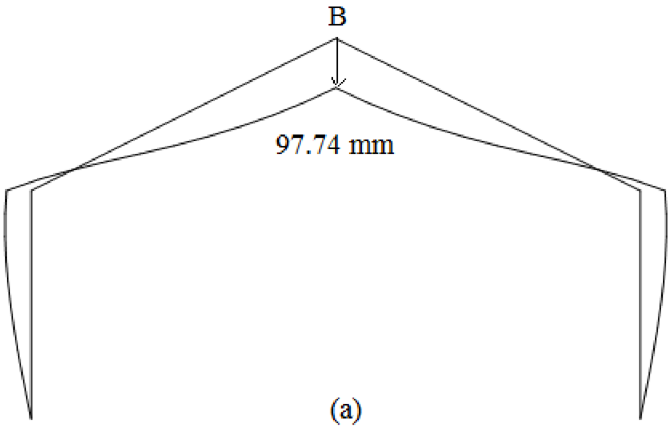
Take E=200 GPa, $I = 400(10)^6$ mm^4, A=100000 mm^2 and $A_C = 1200$ mm^2, 2400 mm^2, 4800 mm^2. Use a Computer software system.


50 kN
50 kN
50 kN
B
I
I
6 m
9 m
I
I
6 m
6 m
6 m
6 m
50 kN
50 kN
50 kN
B
I
I
6 m
Hinge
A_c
Hinge
9 m
I
I
6 m
6 m
6 m
6 m

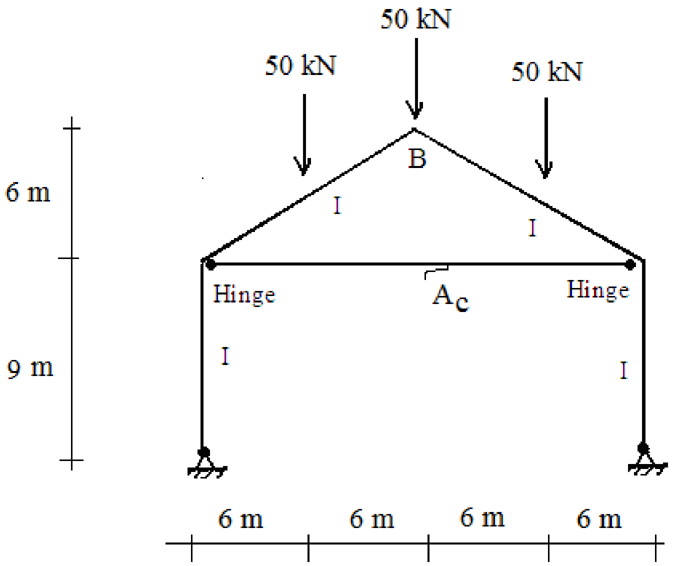
(a)



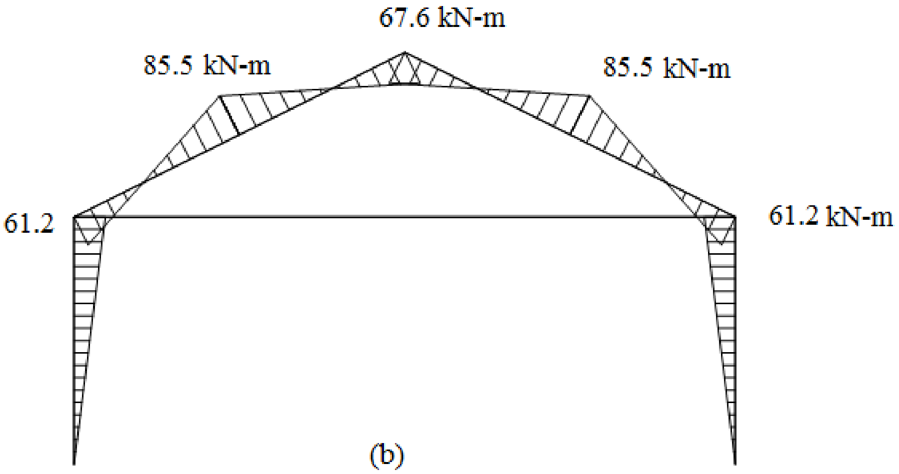
(a)



(a)

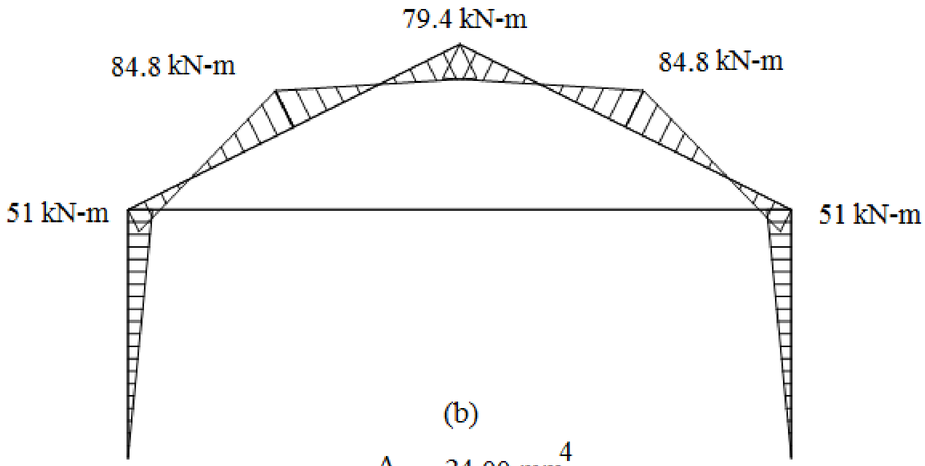


(b)



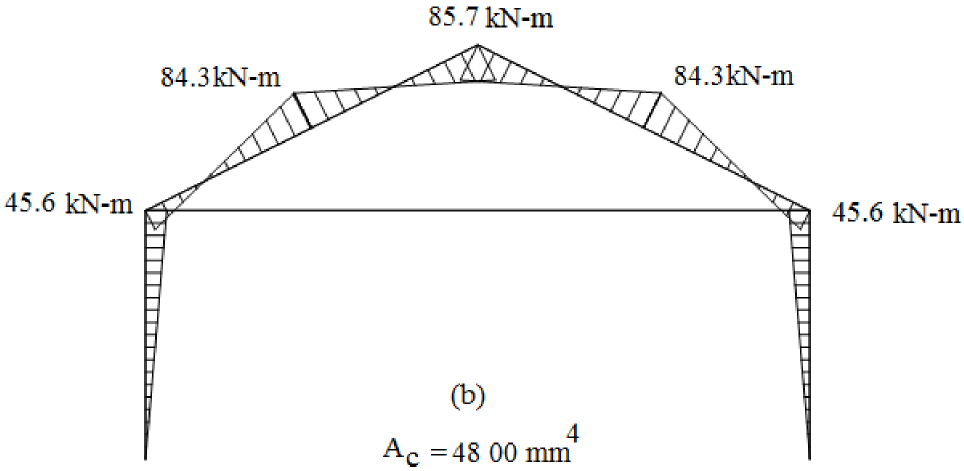
(b)

A_c =1200 mm^4



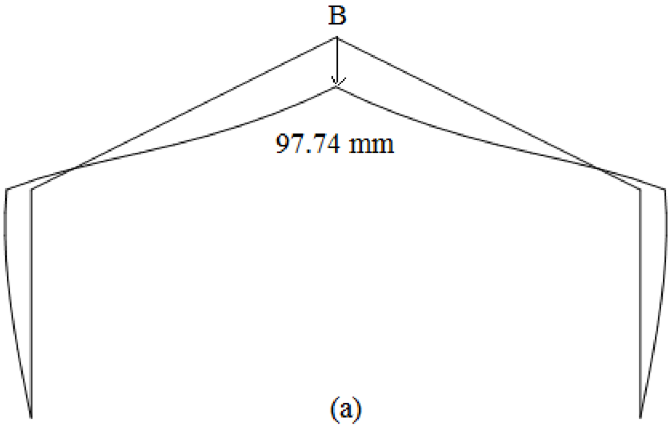
(b)

$A_c = 24\ 00\ \text{mm}^4$

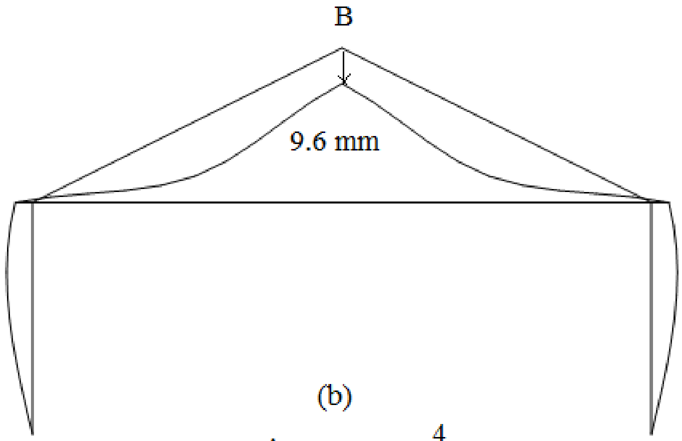


(b)

$A_c = 48\ 00\ \text{mm}^4$

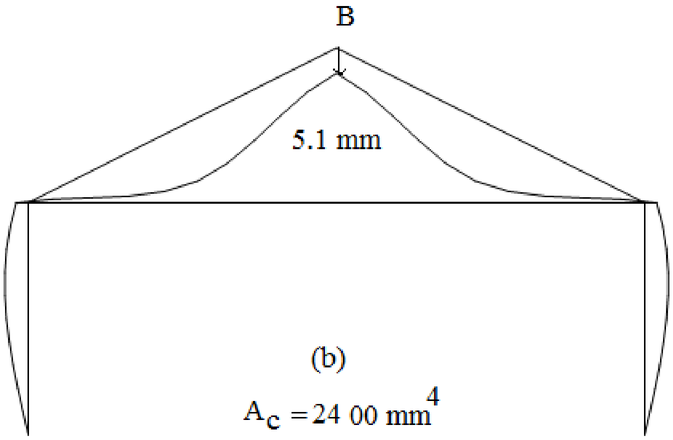


(a)



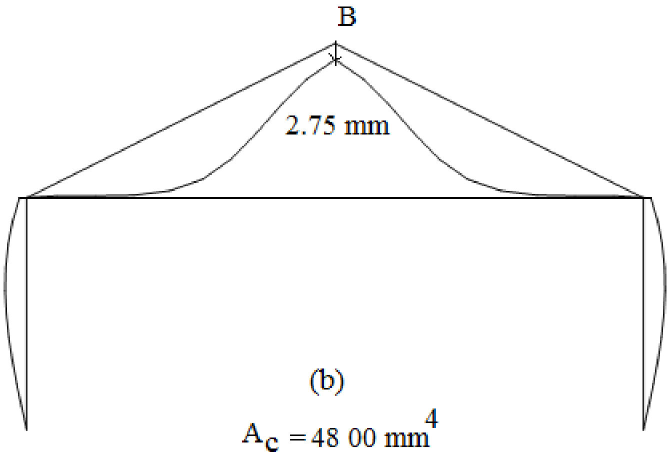
(b)

A_c =1200 mm^4



(b)

A_c =24 00 mm^4



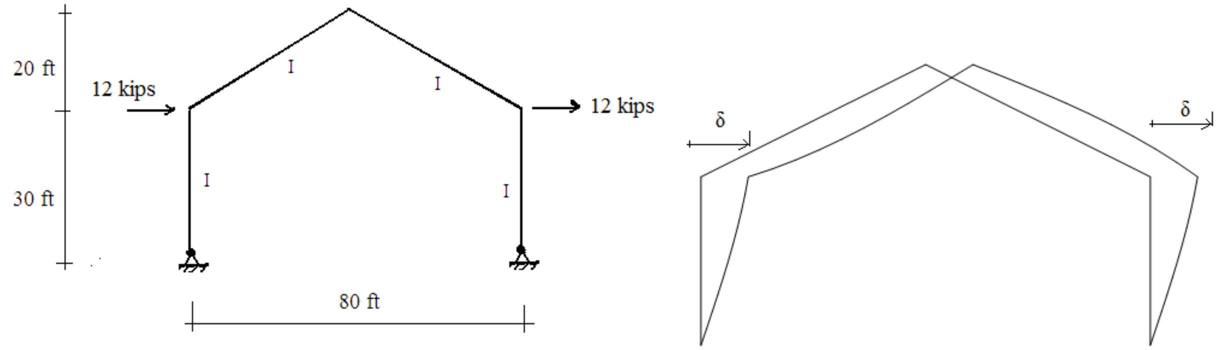
(b)

A_c =48 00 mm^4

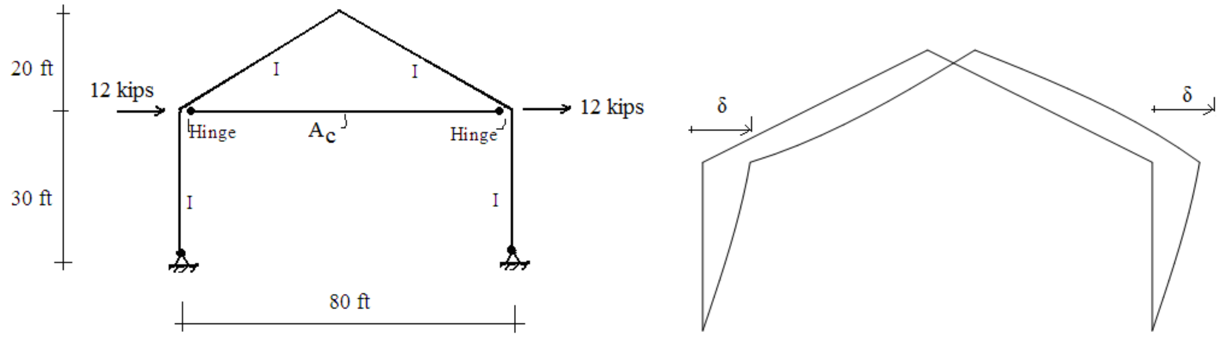
Problem 9.17

Both cases (a) and (b) will are rigid frames and will have large horizontal displacements and will behave the same. In case (b) the added member will not carry any load. In case (c) the diagonal member changes the behavior. The frame will behave like a truss. All the members only carry axial forces and lateral movement will be very small.

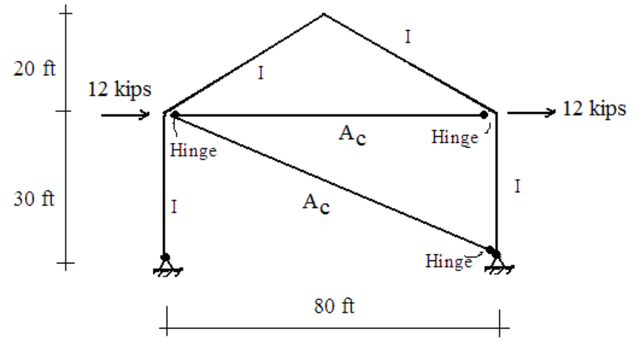
Case (a):



Case (b):

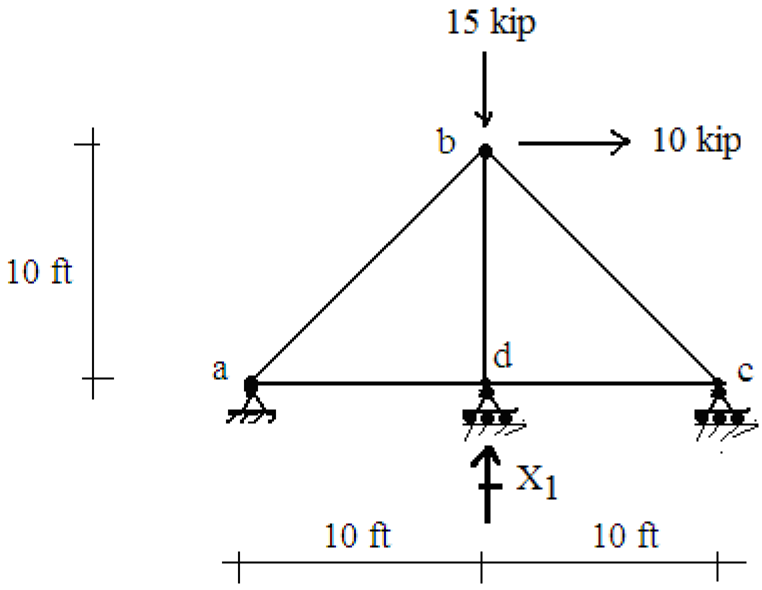


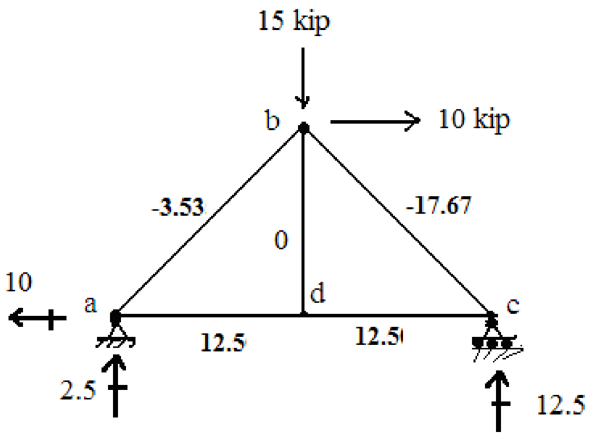
Case (c):



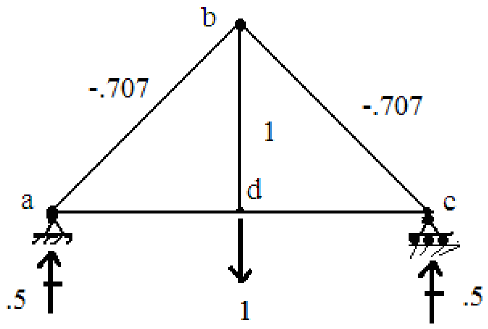
Problem 9.18

E= 29,000 ksi, A= 1 in^2 all members





F_0



δF

member	L, in	F_0	δF	$F\,\delta F\,L$	$\delta F^2\,L$	$F_0 + X_1\,\delta F$
ab	14.14(12)	-3.53	-.707	423.47	84.81	4.62
bc	14.14(12)	-17.67	-.707	2119.75	84.81	-9.51
cd	10(12)	12.5	.5	750	30	6.73
da	10(12)	12.5	.5	750	30	6.73
bd	10(12)	0	1	0	120	-11.53

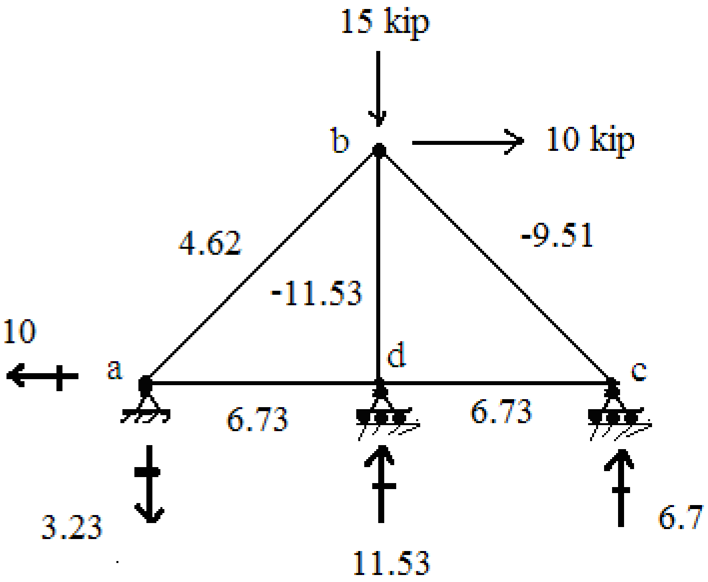
$\sum F\,\delta F\,L = 4043$ $\sum \delta F^2\,L = 349.6$

$$\Delta_{1,0} = \sum F_0\ \delta F \frac{L}{AE} = \frac{4043}{29000} = 0.139$$

$$\delta_{11} = \sum (\delta F)^2 \frac{L}{AE} = \frac{349.6}{29000} = 0.012$$

$$\Delta_{1,0} + \ \delta_{11}\ X_1 = 0 \qquad \rightarrow \qquad X_1 = -\frac{\Delta_{1,0}}{\delta_{11}} = -\frac{0.139}{0.012} = -11.53$$

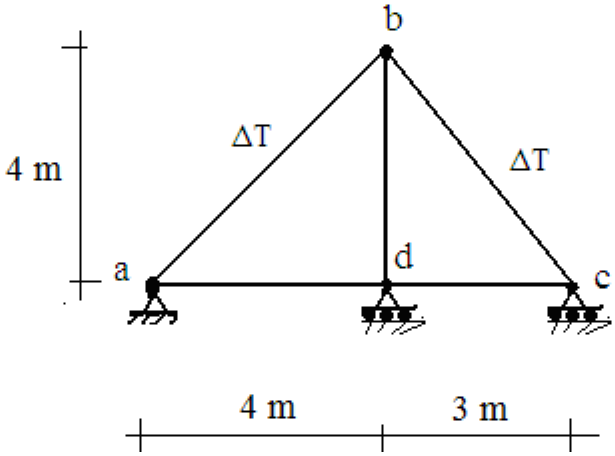
Knowing the value of X_1, we determine the member forces and reactions by using superposition. Member forces are listed below.

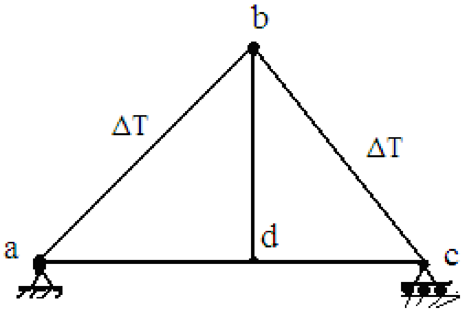


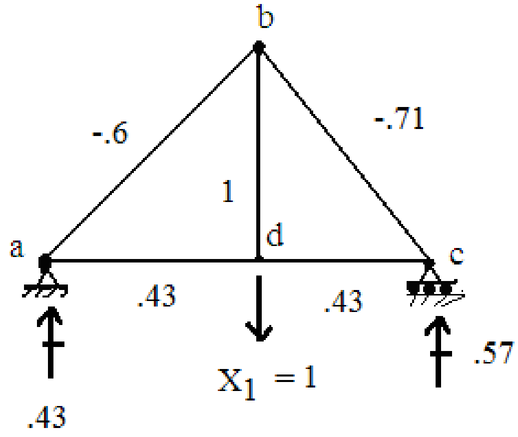
Problem 9.19

Assume the vertical reaction at d as the force redundant.

E= 200 GPa, A= 660 mm^2 all members, α= $12x10^{-6}/\ {}^0C$, $\Delta T = 10^0$ C







member	L, mm	δF	e_{temp} = **α ΔTL**	$\delta F\ e_{temp}$	$\delta F^2\ L$	$X_1\ \delta F$
ab	5657	-.606	0.67884	-.4073	2077.4	-6.9
bc	5657	-.714	0.67884	-.4819	2883.9	-8.1
cd	3000	.428	0	0	549.5	4.8
da	4000	.428	0	0	732.7	4.8
bd	4000	1	0	0	4000	11.4

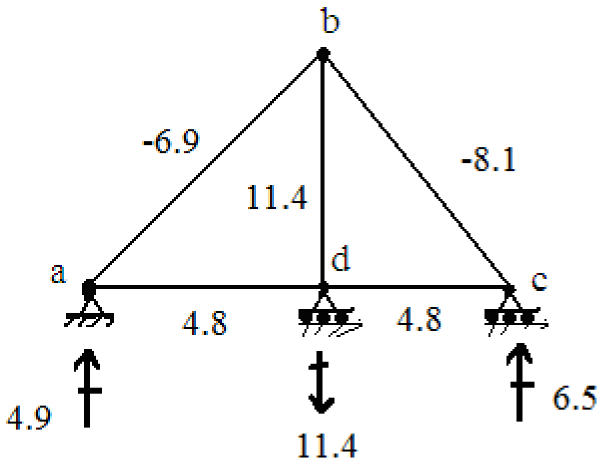
$$\sum e_{temp}\ \delta F = -0.889 \quad \sum L\ (\delta F)^2 = 10243.5$$

$$\Delta_{1,0} = \sum e_{temp}\ \delta F = 0.889$$

$$\delta_{11} = \sum (\delta F)^2 \frac{L}{AE} = \frac{10243.5}{660(200)} = 0.0776$$

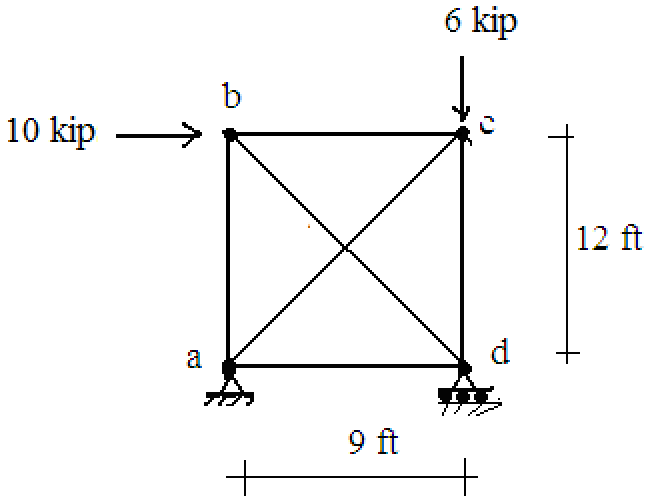
$$\Delta_{1,0} + \delta_{11} X_1 = 0 \qquad \rightarrow \qquad X_1 = -\frac{\Delta_{1,0}}{\delta_{11}} = \frac{0.889}{0.0776} = 11.4$$

Knowing the value of X_1, we determine the member forces and reactions by using superposition. Member forces are listed below.



Problem 9.20

E=29,000 ksi and A= 1 in^2 all members.



member	L, in	F_0	δF	δF^2 L	F_0 δF L	$F_0 + X_1\ \delta F$
ab	12(12)	0	-.8	92.16	0	7.6
bc	9(12)	-10	-.6	38.88	648	-4.3
cd	12(12)	-19.33	-.8	92.16	2226.8	-11.7
da	9(12)	0	-.6	32.88	0	5.7
ac	15(12)	16.66	1	180	2998.8	7.2
bd	15(12)	0	1	180	0	-9.5

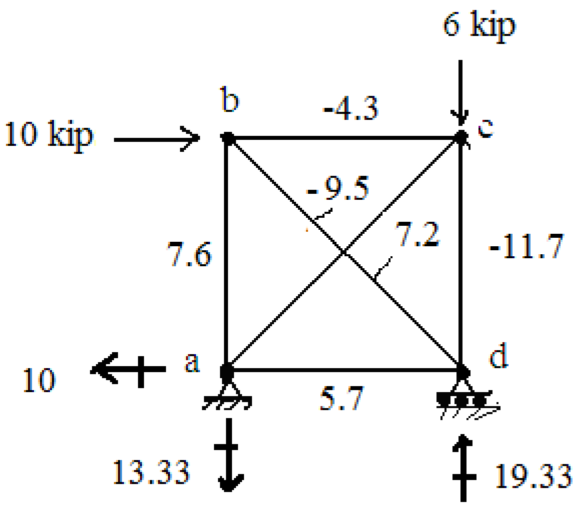
$\sum \delta F^2 L = 616$ $\sum F_0\ \delta FL = 58736$

$$\Delta_{1,0} = \sum F_0\ \delta F \frac{L}{AE} = \frac{5873.6}{2900(1)} = 0.2025$$

$$\delta_{11} = \sum (\delta F)^2 \frac{L}{AE} = \frac{616}{2900(1)} = 0.0212$$

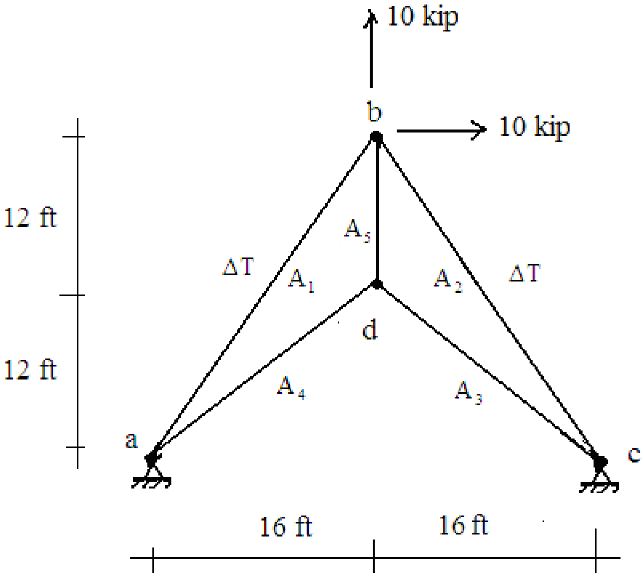
$$\Delta_{1,0} + \ \delta_{11}\ X_1 = 0 \qquad \rightarrow \qquad X_1 = -\frac{\Delta_{1,0}}{\delta_{11}} = -\frac{.2025}{.0212} = -9.5$$

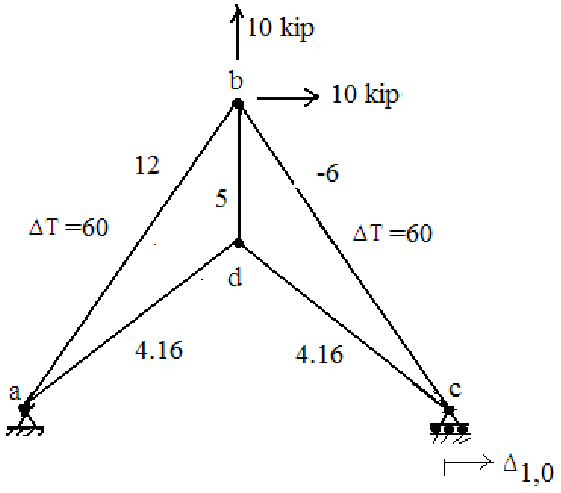
Knowing the value of X_1**,** we determine the member forces and reactions by using superposition. Member forces are listed below.

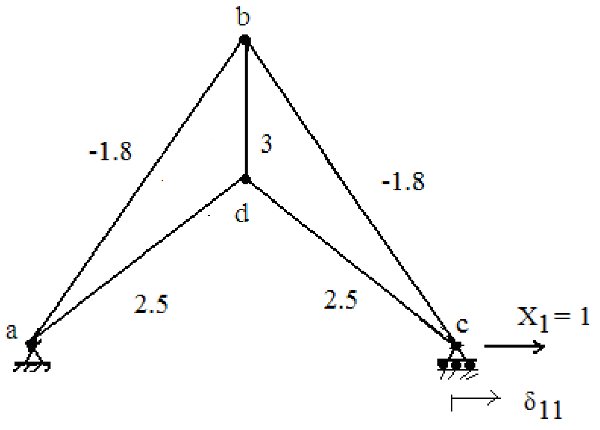


Problem 9.21

$A_1 = A_2 = A_3 = A_4 = 10$ in^2, $A_5 = 5$ in^2, $\alpha = 6.5\text{x}10^{-6}/\ ^0\text{F}$, $\Delta T = 60^0$, E= 29,000 ksi







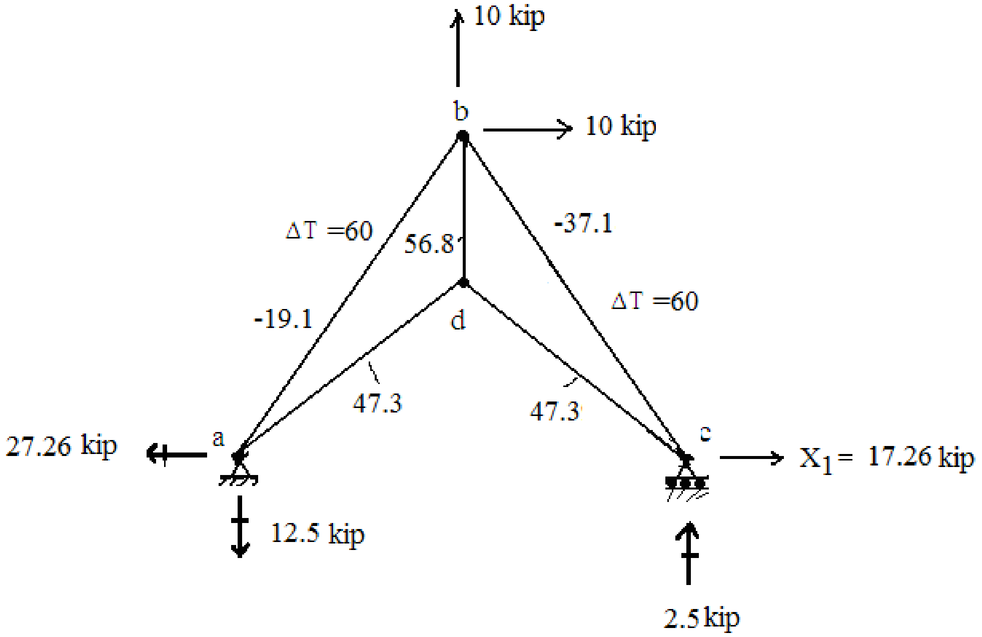
member	L, in	**L/AE**	e_{temp} = **α ΔTL**	F_0	δF	$e_{temp}\ \delta F$	$\frac{F_0\ \delta FL}{AE}$	$\frac{(\delta F)^2\ L}{AE}$	$F_0 + X_1\ \delta F$
ab	28.84(12)	0.001193	0.13497	12	-1.8	-.2429	-.0257	.00386	-19.1
bc	28.84(12)	0.001193	0.13497	-6	-1.8	-.2429	.0128	.00386	-37.1
cd	20(12)	0.000827	0.0936	4.16	2.5	0	.0086	.00516	47.3
da	20(12)	0.000827	0.0936	4.16	2.5	0	.0086	.00516	47.3
bd	12(12)	0.000993	0.05616	5	3	0	.0148	.00893	56.8

$$\Delta_{1,0} = \sum \frac{F_0\ \delta FL}{AE} + \sum e_{temp}\ \delta F = .019 - 0.485 = -0.466$$

$$\delta_{11} = \sum \frac{(\delta F)^2\ L}{AE} = 0.027$$

$$\Delta_{1,0} + \delta_{11}\ X_1 = 0 \qquad \rightarrow \qquad X_1 = -\frac{\Delta_{1,0}}{\delta_{11}} = \frac{0.466}{0.027} = 17.26$$

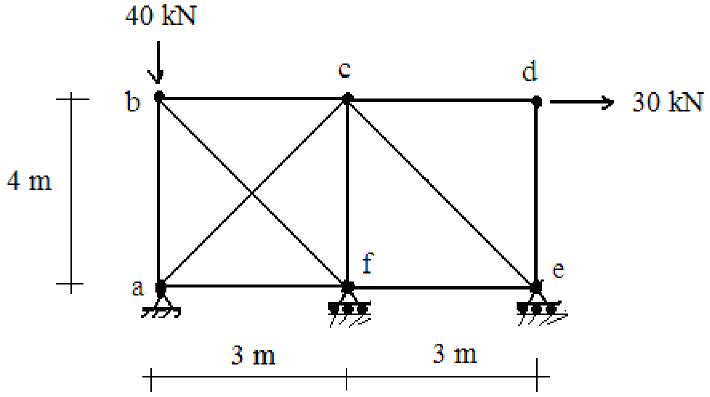
Knowing the value of X_1, we determine the member forces and reactions by using superposition. Member forces are listed below.

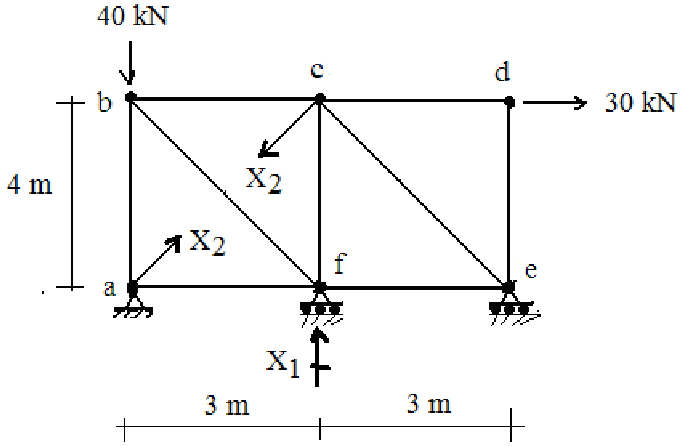


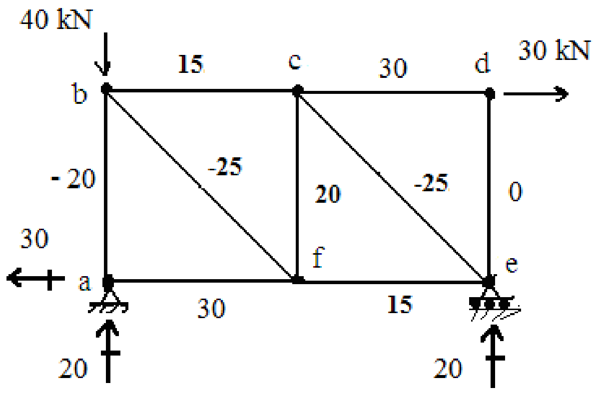
Problem 9.22

Assume the force in member ac and the reaction at support f as force redundants.

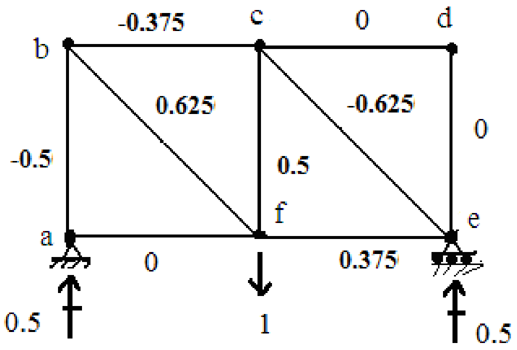
A=1000 mm^2 and E=200 GPa for all the members



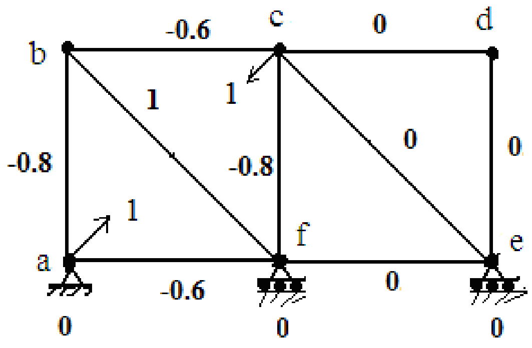




F_0



δF_1



δF_2

member	L, mm	F_0	δF_1	δF_2	$F_0\ \delta F_1\ L$	$F_0\ \delta F_2\ L$	$(\delta F_1)^2\ L$	$(\delta F_2)^2\ L$	$\delta F_1\ \delta F_2\ L$
ab	4000	-20	-.5	-.8	40000	64000	1000	2560	1000
bc	3000	15	-.375	-.6	-16875	-27000	422	1080	675
cd	3000	30	0	0	0	0	0	0	0
de	4000	0	0	0	0	0	0	0	0
ef	3000	15	.375	0	16875	0	422	0	0
fa	3000	30	0	-.6	0	-54000	0	1080	0
bf	5000	-25	.625	1	78125	-125000	1953	5000	3125
ac	5000	0	0	1	0	0	0	5000	0
cf	4000	20	.5	-.8	40000	-64000	1000	2560	-1000
ce	5000	-25	-.625	0	-78125	0	1953	0	0

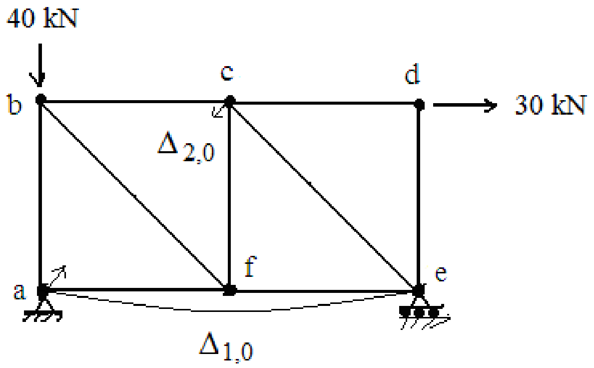
$\sum F_0\delta F_1 L = 80000$ $\sum F_0\delta F_2 L = -206000$

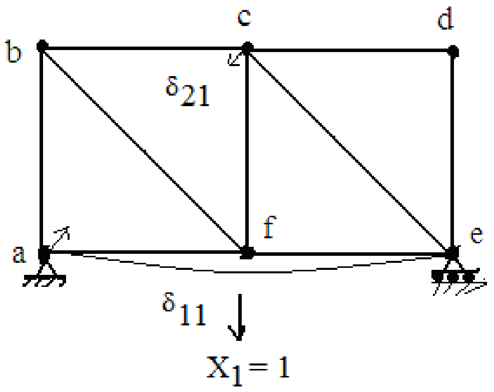
$\sum (\delta F_1)^2 L = 6750$ $\sum (\delta F_2)^2 L = 17280$ $\sum \delta F_1\ \delta F_2 L = 3800$

$$\Delta_{1,0} = \sum \frac{F_0 \delta F_1 L}{EA} = \frac{80000}{200(1000)} = .4 \text{ mm} \qquad \Delta_{2,0} = \sum \frac{F_0 \delta F_2 L}{EA} = -\frac{206000}{200(1000)} = \text{ -1.03 mm}$$

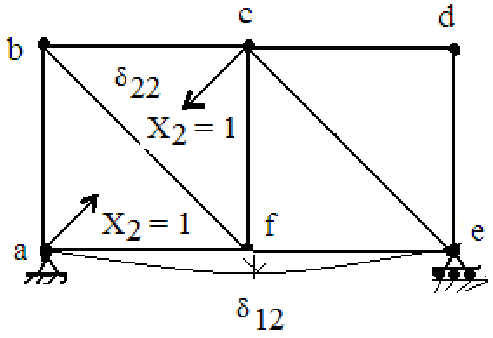
$$\delta_{11} = \sum \frac{(\delta F_1)^2 L}{EA} = \frac{6750}{200(1000)} = 0.03375 \qquad \delta_{22} = \sum \frac{(\delta F_2)^2 L}{EA} = \frac{17280}{200(1000)} = 0.0864$$

$$\delta_{12} = \delta_{21} = \sum \frac{\delta F_1 \delta F_2 L}{EA} = \frac{3800}{200(1000)} = 0.019$$





δF₁



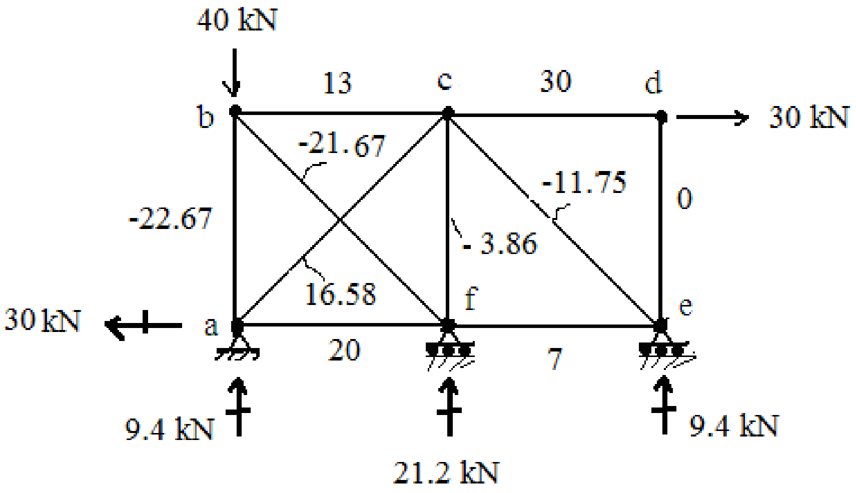
δF₂

$$\begin{aligned} \Delta_{1,0} + \delta_{11} X_1 + \delta_{12} X_2 &= 0 \\ \Delta_{2,0} + \delta_{21} X_1 + \delta_{22} X_2 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{cases} X_1 = -21.2 \text{ kN} \\ X_2 = 16.58 \text{ kN} \end{cases}$$

Knowing the value of X_1 and X_2, we determine the member forces and reactions by using superposition.

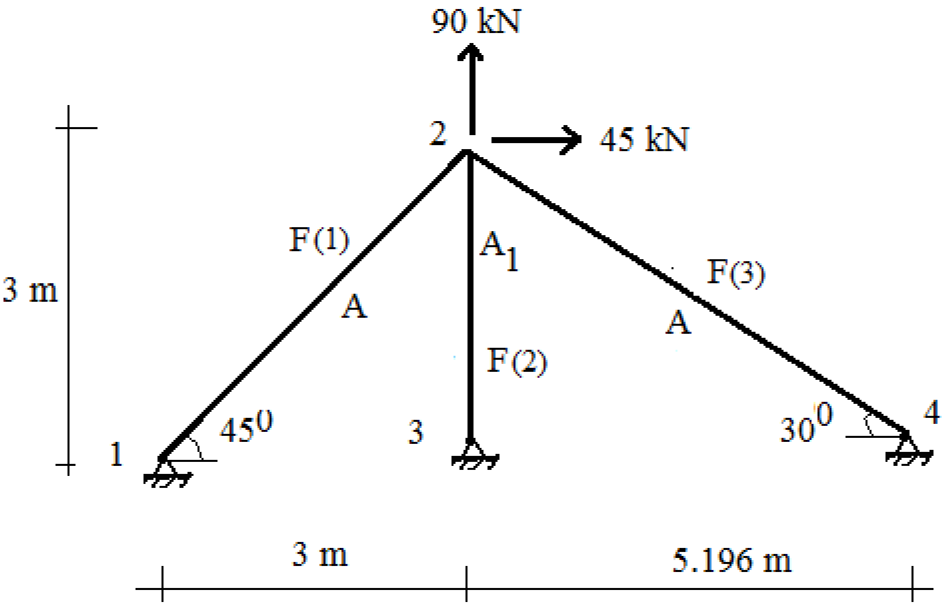
member	F_0	δF_1	δF_2	$F_0 + X_1\ \delta F_1 + X_2\ \delta F_2$
ab	-20	-.5	-.8	-22.67
bc	15	-.375	-.6	13.0
cd	30	0	0	30
de	0	0	0	0
ef	15	.375	0	7
fa	30	0	-.6	20
bf	-25	.625	1	-21.67
ac	0	0	1	16.58
cf	20	.5	-.8	-3.86
ce	-25	-.625	0	-11.75

Member forces are listed below.



Problem 10.1

Take E = 200 GPa and $A = 2000\ \text{mm}^2$



(a) $A_1 = \dfrac{1}{2}A$

$$\frac{AE}{L_1} = \frac{(2000)200}{3\sqrt{2}(1000)} = 94.28\ \ \frac{kN}{mm}$$

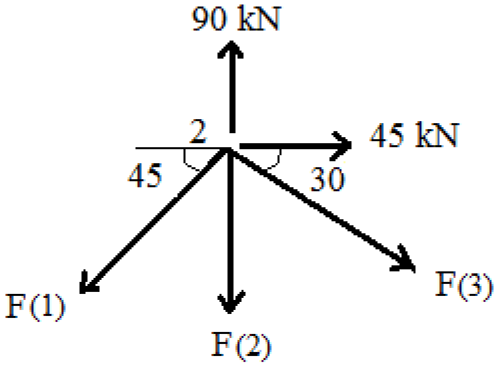
$$\frac{A_1E}{L_2} = \frac{(1000)200}{3(1000)} = 66.67\ \ \frac{kN}{mm}$$

$$\frac{AE}{L_3} = \frac{(2000)200}{6(1000)} = 66.67\ \ \frac{kN}{mm}$$

$$F_{(1)} = \frac{AE}{L_1}\cos 45\ u_2 + \frac{AE}{L_1}\sin 45\ v_2 = 66.67u_2 + 66.67v_2$$

$$F_{(2)} = \frac{A_1E}{L_2}\ v_2 = 66.67v_2$$

$$F_{(3)} = -\frac{AE}{L_3}\cos 30\ u_2 + \frac{AE}{L_3}\sin 30\ v_2 = -57.73u_2 + 33.33v_2$$



$$\sum F_x = 0 \rightarrow \qquad -\ F_{(1)}\cos 45 \ +\ F_{(3)}\cos 30 \ +\ 45 = 0$$

$$\sum F_y = 0 \uparrow \qquad -\ F_{(1)}\sin 45 \ -\ F_{(2)} - F_{(3)}\sin 30 \ +\ 90 = 0$$

$$\Downarrow$$

$$\begin{aligned} 0.707\ F_{(1)} \ -\ 0.86\ F_{(3)} \ &= \ 45 \\ 0.707\ F_{(1)} \ +\ F_{(2)} +\ 0.5\ F_{(3)} \ &= \ 90 \end{aligned}$$

$$\Downarrow$$

$$\begin{aligned} 0.707\ \left(66.67\mathrm{u}_2 + 66.67v_2\right) \ -\ 0.86\ \left(-57.73\mathrm{u}_2 + 33.33v_2\right) \ &= \ 45 \\ 0.707\ \left(66.67\mathrm{u}_2 + 66.67v_2\right) \ +\ 66.67v_2 +\ 0.5\ \left(-57.73\mathrm{u}_2 + 33.33v_2\right) \ &= \ 90 \end{aligned}$$

$$\Downarrow$$

$$\begin{aligned} 96.78\ \mathrm{u}_2 + 18.47\ v_2 &= 45 \\ 18.27\ \mathrm{u}_2 + 130.47\ v_2 &= 90 \end{aligned} \quad \Rightarrow \quad \begin{cases} \mathrm{u}_2 = 0.342\ \ \mathrm{mm} \rightarrow \\ v_2 = 0.645\ \ \mathrm{mm} \uparrow \end{cases}$$

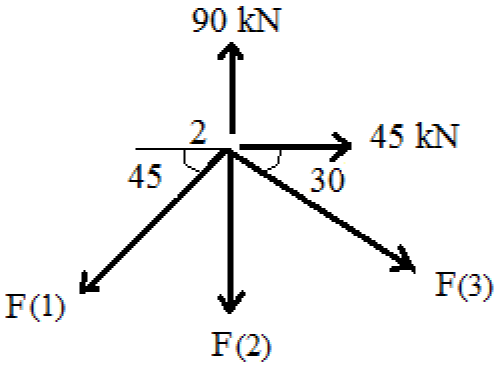
Next, we substitute for u_2 and v_2 to determine member forces

$$\begin{aligned} F_{(1)} &= 66.67\mathrm{u}_2 + 66.67v_2 = 65.8\ \ \mathrm{kN} \\ F_{(2)} &= 66.67v_2 = 43\ \ \mathrm{kN} \\ F_{(3)} &= -57.73\mathrm{u}_2 + 33.33v_2 = 1.75\ \ \mathrm{kN} \end{aligned}$$

(b) $A_1 = 2A$

$$\frac{AE}{L_1} = \frac{(2000)200}{3\sqrt{2}(1000)} = 94.28 \ \frac{kN}{mm}$$
$$\frac{A_1E}{L_2} = \frac{(4000)200}{3(1000)} = 266.67 \ \frac{kN}{mm}$$
$$\frac{AE}{L_3} = \frac{(2000)200}{6(1000)} = 66.67 \ \frac{kN}{mm}$$

$$F_{(1)} = \frac{AE}{L_1}\cos 45 \ u_2 + \frac{AE}{L_1}\sin 45 \ v_2 = 66.67u_2 + 66.67v_2$$
$$F_{(2)} = \frac{A_1E}{L_2} \ v_2 = 266.67v_2$$
$$F_{(3)} = -\frac{AE}{L_3}\cos 30 \ u_2 + \frac{AE}{L_3}\sin 30 \ v_2 = -57.73u_2 + 33.33v_2$$



$$\sum F_x = 0 \rightarrow \qquad 0.707 \ F_{(1)} - 0.86 \ F_{(3)} = 45$$

$$\sum F_y = 0 \uparrow \qquad 0.707 \ F_{(1)} + F_{(2)} + 0.5 \ F_{(3)} = 90$$

$$\Downarrow$$

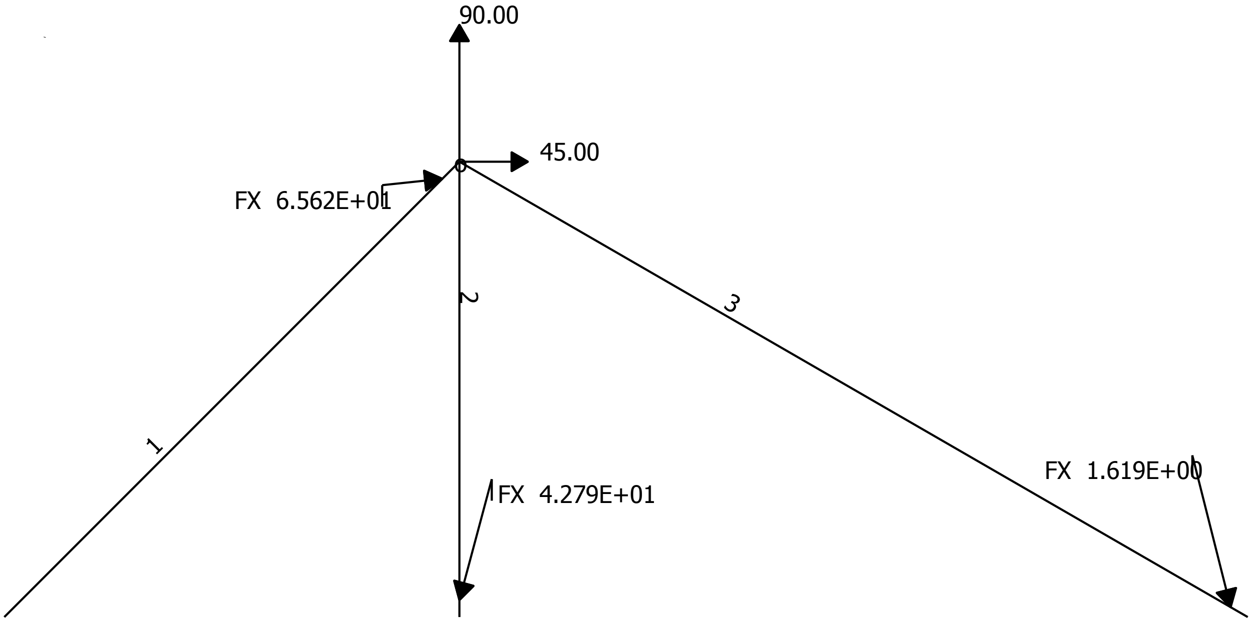
$$\begin{matrix} 96.78 \ u_2 + 18.47 \ v_2 = 45 \\ 18.27 \ u_2 + 330.47 \ v_2 = 90 \end{matrix} \quad \Rightarrow \quad \begin{cases} u_2 = 0.417 \ mm \rightarrow \\ v_2 = 0.25 \ mm \uparrow \end{cases}$$

Next, we substitute for u_2 and v_2 to determine member forces

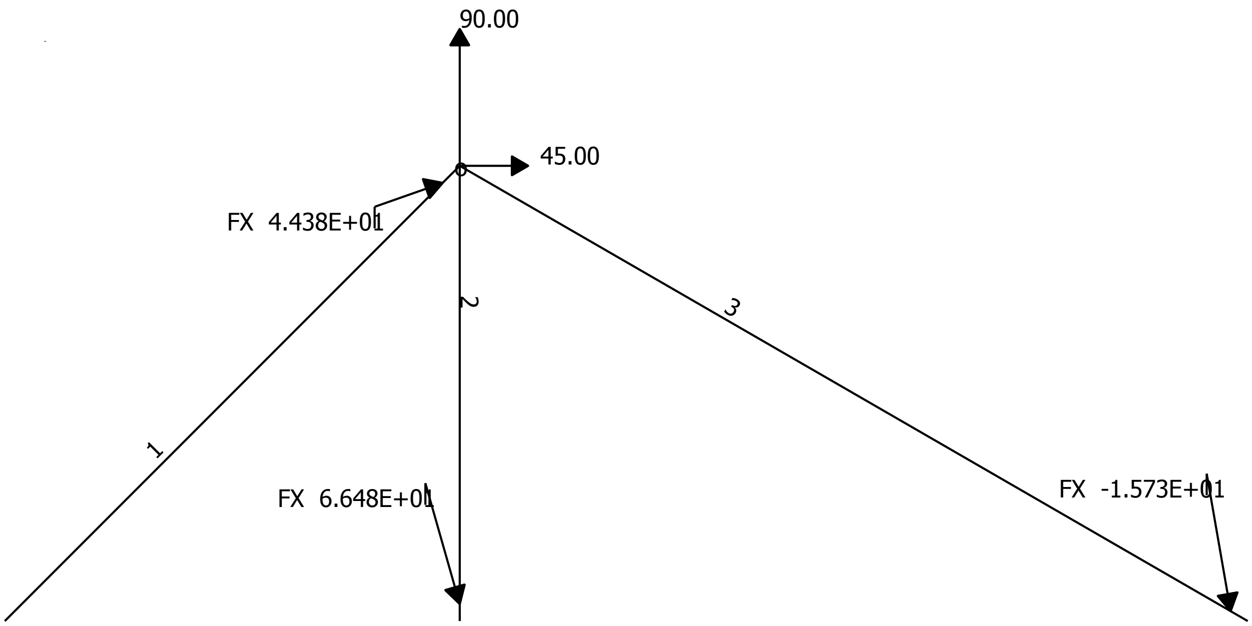
$$F_{(1)} = 66.67u_2 + 66.67v_2 = 44.47 \ kN$$
$$F_{(2)} = 266.67v_2 = 66.67 \ kN$$
$$F_{(3)} = -57.73u_2 + 33.33v_2 = -15.74 \ kN$$

(c) Results based on computer based analysis listed below

$$\text{For } A_1 = \frac{1}{2}A \begin{cases} F_{(1)} = 65.62 \text{ kN} \\ F_{(2)} = 42.79 \text{ kN} \\ F_{(3)} = \ 1.62 \text{ kN} \end{cases} \qquad \begin{cases} u_2 = 0.342 \text{ mm} \rightarrow \\ v_2 = 0.642 \text{ mm} \uparrow \end{cases}$$

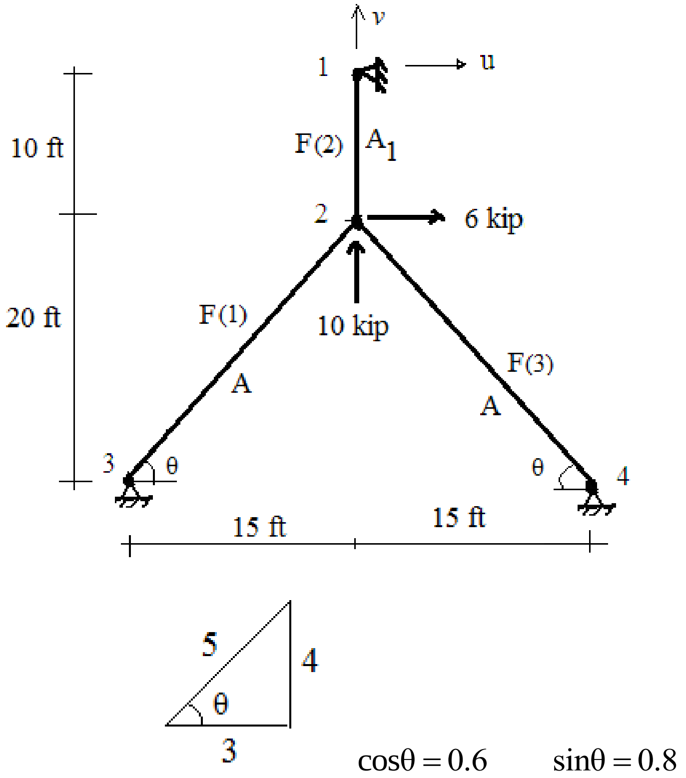


$$\text{For } A_1 = 2A \qquad \begin{cases} F_{(1)} = 44.37 \text{ kN} \\ F_{(2)} = 66.48 \text{ kN} \\ F_{(3)} = \text{-}15.73 \text{ kN} \end{cases} \qquad \begin{cases} u_2 = 0.416 \text{ mm} \rightarrow \\ v_2 = 0.249 \text{ mm} \uparrow \end{cases}$$



Problem 10.2

Take $A=0.1\text{ in}^2$, $A_1=0.4\text{ in}^2$, and E=29,000 ksi.



$\cos\theta = 0.6 \qquad \sin\theta = 0.8$

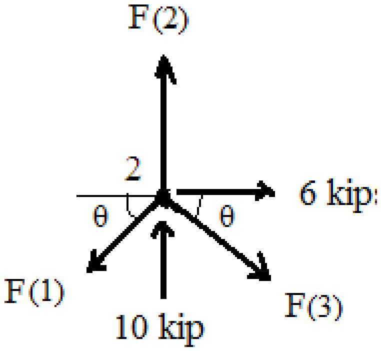
$$\frac{AE}{L_1}=\frac{AE}{L_3}=\frac{(0.1)29000}{25(12)}=9.67\ \frac{\text{kip}}{\text{in}} \qquad \frac{A_1E}{L_2}=\frac{(0.4)29000}{10(12)}=96.67\ \frac{\text{kip}}{\text{in}}$$

(a) The loading shown

$$F_{(1)}=\frac{AE}{L_1}\cos\theta\ u_2+\frac{AE}{L_1}\sin\theta\ v_2=5.8\ u_2+7.74\ v_2$$

$$F_{(2)}=-\frac{A_1E}{L_2}\ v_2=-96.67\ v_2$$

$$F_{(3)}=-\frac{AE}{L_3}\cos\theta\ u_2+\frac{AE}{L_3}\sin\theta\ v_2=-5.8\ u_2+7.74\ v_2$$



$$\sum F_x=0\rightarrow \qquad -0.6\ F_{(1)}+\ 0.6\ F_{(3)}+6=0$$

$$\sum F_y=0\uparrow \qquad -0.8\ F_{(1)}+\ F_{(2)}-0.8\ F_{(3)}+10=0$$

$$\Downarrow$$

$$\begin{matrix}6.96\ u_2=6\\ 109\ v_2=10\end{matrix}\quad\Rightarrow\quad\begin{cases}u_2=0.862\ \text{in}\rightarrow\\ v_2=0.092\ \text{in}\uparrow\end{cases}$$

Next, we substitute for u_2 and v_2 to determine the member forces

$F_{(1)}=5.8\ u_2+7.74\ v_2=5.7\ \text{kip}$

$F_{(2)}=-96.67\ v_2=-\ 8.89\ \text{kip}$

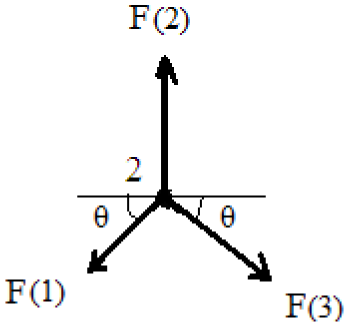
$F_{(3)}=-5.8\ u_2+7.74\ v_2=-\ 4.3\ \text{kip}$

(b) Support #1 moves as follows: $u=\frac{1}{8}\text{inch}\rightarrow$ and $v=\frac{1}{2}\text{inch}\uparrow$

$$F_{(1)} = \frac{AE}{L_1}\cos\theta\ u_2 + \frac{AE}{L_1}\sin\theta\ v_2 = 5.8\ u_2 + 7.74\ v_2$$

$$F_{(2)} = -\frac{A_1E}{L_2}\ (v_2 - \frac{1}{2}) = -96.67\ (v_2 - \frac{1}{2})$$

$$F_{(3)} = -\frac{AE}{L_3}\cos\theta\ u_2 + \frac{AE}{L_3}\sin\theta\ v_2 = -5.8\ u_2 + 7.74\ v_2$$



$$\sum F_x = 0 \qquad F_{(1)} = F_{(3)} \qquad \Rightarrow \qquad u_2 = 0$$

$$\sum F_y = 0 \uparrow \qquad 0.8\ (F_{(1)} + F_{(3)}) - F_{(2)} = 0$$

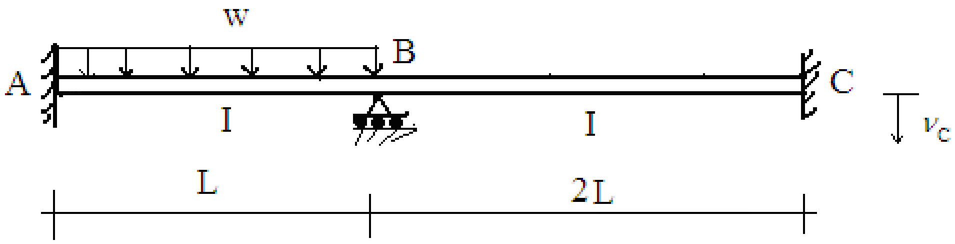
$$0.8\ (15.48\ v_2) + 96.67\ (v_2 - \frac{1}{2}) = 0 \qquad \Rightarrow \qquad v_2 = 0.443\ \text{in} \uparrow$$

$$\therefore \quad \begin{aligned} F_{(1)} &= F_{(3)} = 3.4\ \text{kip} \\ F_{(2)} &= 5.5\ \text{kip} \end{aligned}$$

For the following beams and frames defined in Problems 10.3- 10.18, determine the member end moments using the slope-deflection equations.

Problem 10.3

Assume E=29,000 ksi, $I = 200\ \text{in}^4$, $L = 30\ \text{ft}$, $v_C = .6\ \text{in} \downarrow$ and $w = 1.2\ \text{kip/ft}$.

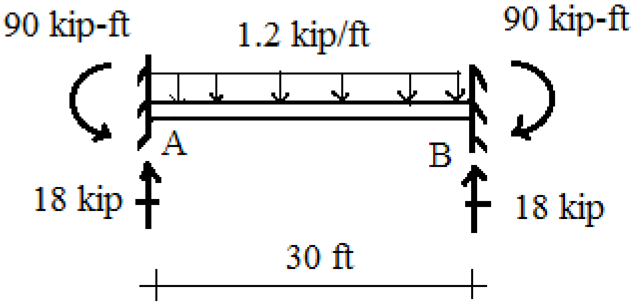


Let $k_{\text{member AB}} = \dfrac{EI}{L} = 2k \qquad k_{\text{member BC}} = \dfrac{EI}{2L} = k$

$$\rho_{AB} = \frac{\nu_B - \nu_A}{L} = 0$$

$$\rho_{BC} = \frac{\nu_C - \nu_B}{2L} = \frac{-0.6}{60(12)} = -\frac{1}{1200}$$

$$k = \frac{EI}{2L} = \frac{29000(200)}{2(30)}\frac{1}{(144)} = 671.3 \text{ kip-ft}$$



The slope-deflection equations take the form:

$$M_{AB} = 4k\left\{\ \theta_B\ \right\} + 90$$

$$M_{BA} = 4k\left\{2\theta_B\ \right\} -90$$

$$M_{BC} = 2k\left\{2\theta_B - 3(\frac{\nu_C - \nu_B}{2L})\right\} = 2k\left\{2\theta_B - 3\rho_{BC}\right\}$$

$$M_{CB} = 2k\left\{\theta_B - 3(\frac{\nu_C - \nu_B}{2L})\right\} = 2k\left\{\theta_B - 3\rho_{BC}\right\}$$

Enforce moment equilibrium at node B:

$$M_{BA} + M_{BC} = 0$$

$$\Downarrow$$

$$8k\theta_B - 90 + 2k\left\{2\theta_B - 3\rho_{BC}\right\} = 0$$

$$\Downarrow$$

$$12k\theta_B - 6k\,\rho_{BC} = 90$$

$$\Downarrow$$

$$\theta_B = \frac{90 + 6(671.3)(-\frac{1}{1200})}{12(671.3)} = 0.01076 \text{ rad} \qquad \text{counterclockwise}$$

The corresponding end moments are:

$$M_{AB} = 4(671.3)\{\ 0.01076\} + 90 = 118.89 \text{ kip-ft}$$

$$M_{BA} = 4(671.3)\{2(0.01076)\ \} - 90 = -32.21 \text{ kip-ft}$$

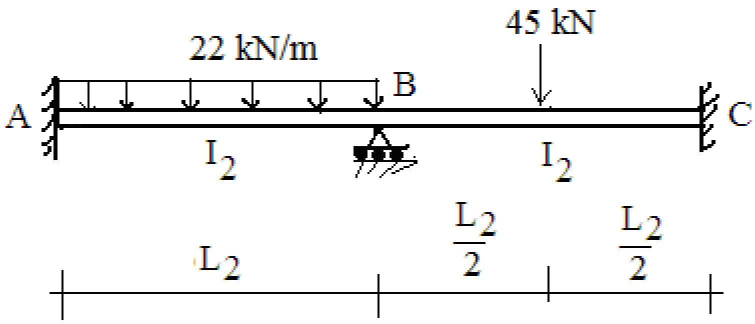
$$M_{BC} = 2(671.3)\left\{2(0.01076)\ -\ 3(-\frac{1}{1200})\right\} = 32.25 \text{ kip-ft}$$

$$M_{CB} = 2(671.3)\left\{0.01076\ -\ 3(-\frac{1}{1200})\right\} = 17.80 \text{ kip-ft}$$

Problem 10.4

Assume E= 200 GPa, $I_2 = 80(10)^6\ mm^4$ and $L_2 = 6\ m$.

(a) Let $k_{\text{member AB}} = \frac{EI_2}{L_2} = k$ $k_{\text{member BC}} = \frac{EI_2}{L_2} = k$



$$M^F_{AB} = \frac{22(6)^2}{12} = 66 \text{ kN-m} \qquad M^F_{BA} = -\frac{22(6)^2}{12} = -66 \text{ kN-m}$$

$$M^F_{BC} = \frac{45(6)}{8} = 33.75 \text{ kN-m} \qquad M^F_{CB} = -\frac{45(6)}{8} = -33.75 \text{ kN-m}$$

The slope-deflection equations take the form:

$$M_{AB} = 2k\ \theta_B + 66$$
$$M_{BA} = 4k\ \theta_B - 66$$

$$M_{BC} = 4k\ \theta_B + 33.75$$
$$M_{CB} = 2k\ \theta_B - 33.75$$

Enforce moment equilibrium at node B:

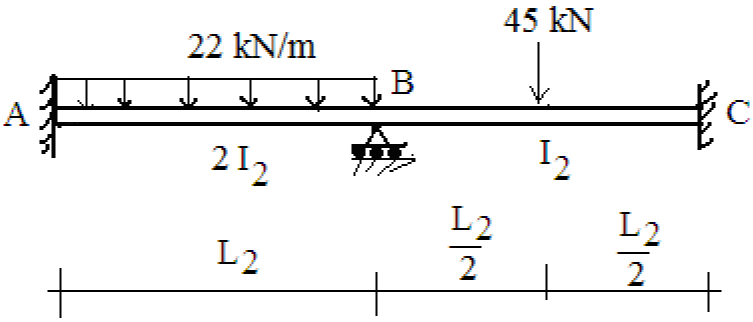
$$M_{BA} + M_{BC} = 0 \qquad 4k\ \theta_B - 66 + 4k\ \theta_B + 33.75 = 0 \qquad \Rightarrow \qquad k\ \theta_B = 4.03$$

The corresponding end moments are:

$$M_{AB} = 2k\ \theta_B + 66 = 74.06\ \text{kN-m}$$
$$M_{BA} = 4k\ \theta_B - 66 = \ \text{-49.9}\ \text{kN-m}$$

$$M_{BC} = 4k\ \theta_B + 33.75 = \ 49.9\ \text{kN-m}$$
$$M_{CB} = 2k\ \theta_B - 33.75 = \ -25.7\ \text{kN-m}$$

(b) Let $k_{\text{member AB}} = \dfrac{E(2I_2)}{L_2} = 2k$ $k_{\text{member BC}} = \dfrac{EI_2}{L_2} = k$



The slope-deflection equations take the form:

$$M_{AB} = 4k\ \theta_B + 66$$
$$M_{BA} = 8k\ \theta_B - 66$$

$$M_{BC} = 4k\ \theta_B + 33.75$$
$$M_{CB} = 2k\ \theta_B - 33.75$$

Enforce moment equilibrium at node B:

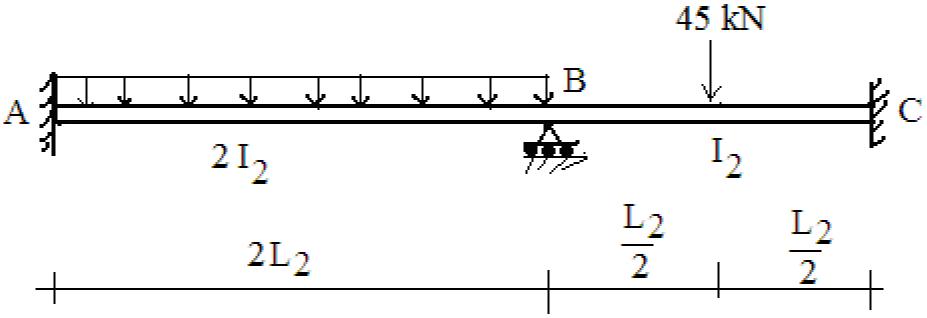
$$M_{BA} + M_{BC} = 0 \qquad 8k\ \theta_B - 66 + 4k\ \theta_B + 33.75 = 0 \qquad \Rightarrow \qquad k\ \theta_B = 2.68$$

The corresponding end moments are:

$$M_{AB} = 4k\ \theta_B + 66 = 76.7\ \text{kN-m}$$
$$M_{BA} = 8k\ \theta_B - 66 = -44.5\ \text{kN-m}$$

$$M_{BC} = 4k\ \theta_B + 33.75 = 44.5\ \text{kN-m}$$
$$M_{CB} = 2k\ \theta_B - 33.75 = -28.4\ \text{kN-m}$$

(c) Let $k_{\text{member AB}} = \dfrac{E(2I_2)}{(2L_2)} = k$ $\qquad k_{\text{member BC}} = \dfrac{EI_2}{L_2} = k$



$$M^F_{AB} = \frac{22(12)^2}{12} = 264\ \text{kN-m} \qquad M^F_{BA} = -\frac{22(12)^2}{12} = -264\ \text{kN-m}$$
$$M^F_{BC} = \frac{45(6)}{8} = 33.75\ \text{kN-m} \qquad M^F_{CB} = -\frac{45(6)}{8} = -33.75\ \text{kN-m}$$

The slope-deflection equations take the form:

$$M_{AB} = 2k\ \theta_B + 264$$
$$M_{BA} = 4k\ \theta_B - 264$$

$$M_{BC} = 4k\ \theta_B + 33.75$$
$$M_{CB} = 2k\ \theta_B - 33.75$$

Enforce moment equilibrium at node B:

$$M_{BA} + M_{BC} = 0 \qquad 4k\ \theta_B - 264 + 4k\ \theta_B + 33.75 = 0 \qquad \Rightarrow \qquad k\ \theta_B = 28.78$$

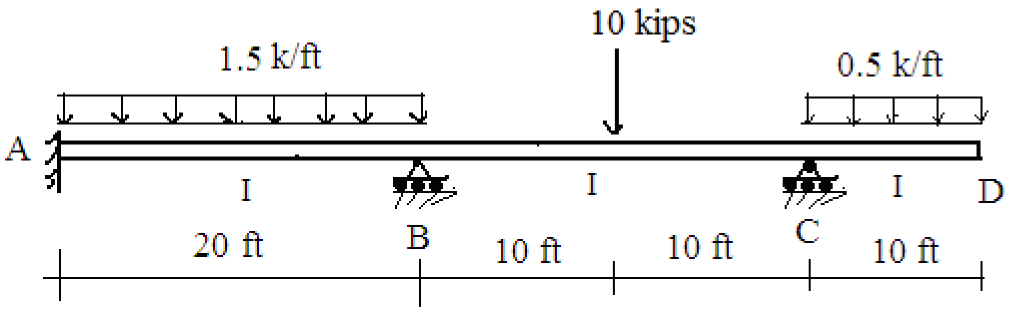
The corresponding end moments are:

$$M_{AB} = 2k\ \theta_B + 264 = 321.5\ \text{kN-m}$$
$$M_{BA} = 4k\ \theta_B - 264 = \text{-148.9 kN-m}$$

$$M_{BC} = 4k\ \theta_B + 33.75 = 148.9\ \text{kN-m}$$
$$M_{CB} = 2k\ \theta_B - 33.75 = 23.8\ \text{kN-m}$$

Problem 10.5

E=29,000 ksi and $I = 300\ \text{in}^4$, L=20 ft



Let $$k_{\text{member AB}} = k_{\text{member BC}} = \frac{EI}{L} = k$$

$$M^F_{AB} = \frac{1.5(20)^2}{12} = 50\ \text{kip-ft} \qquad M^F_{BA} = -\frac{1.5(20)^2}{12} = -50\ \text{kip-ft}$$

$$M^F_{BC} = \frac{10(20)}{8} = 25\ \text{kip-ft} \qquad M^F_{CB} = -\frac{10(20)}{8} = -25\ \text{kip-ft}$$

The slope-deflection equations take the form:

$$M_{AB} = 2k\ \theta_B + 50$$
$$M_{BA} = 4k\ \theta_B - 50$$

$$M_{BC} = 2k\ (2\theta_B + \theta_C) + 25$$
$$M_{CB} = 2k\ (\theta_B + 2\theta_C) - 25$$

Enforce moment equilibrium at nodes B and C:

$$M_{CB} + 25 = 0 \qquad 2k\ (\theta_B + 2\theta_C) - 25 + 25 = 0 \quad \Rightarrow \quad 2\theta_C = -\theta_B$$
$$M_{BA} + M_{BC} = 0 \qquad 4k\ \theta_B - 50 + 2k\ (2\theta_B + \theta_C) + 25 = 0 \quad \Rightarrow \quad 8k\ \theta_B + 2k\theta_C = 25$$

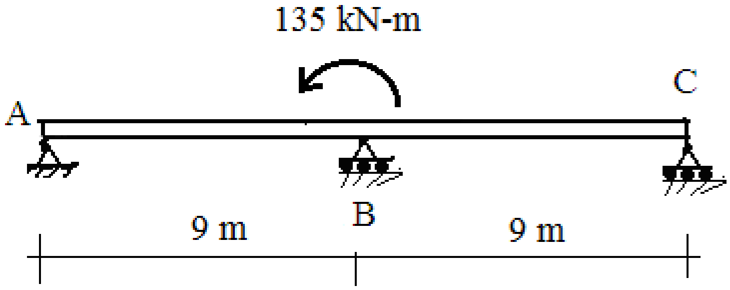
$$\therefore \qquad k\ \theta_B = \frac{25}{7} \qquad k\ \theta_C = -\frac{25}{14}$$

The corresponding end moments are:

$$M_{AB} = 2k\ \theta_B + 50 = 57.1\ \text{kip-ft}$$
$$M_{BA} = 4k\ \theta_B - 50 = -35.7\ \text{kip-ft}$$
$$M_{BC} = 2k\ (2\theta_B + \theta_C) + 25 = 35.7\ \text{kip-ft}$$
$$M_{CB} = 2k\ (\theta_B + 2\theta_C) - 25 = -25\ \text{kip-ft}$$

Problem 10.6

E= 200 GPa, $I = 80(10)^6$ mm^4, P=45kN, h =3 m and $L = 9$ m.



The slope-deflection equations take the form:

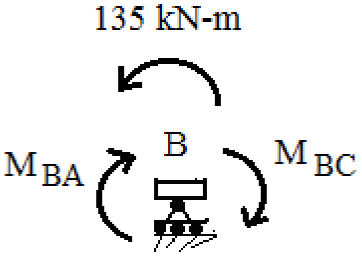
$$M_{AB} = 0$$

$$M_{BA\,\text{modified}} = \frac{3EI}{L}\{\theta_B\}$$

$$M_{CB} = 0$$

$$M_{BC\,\text{modified}} = \frac{3EI}{L}\{\theta_B\}$$

Enforce moment equilibrium at node B:



$$M_{BA\text{modified}} + M_{BC\text{modified}} = 135$$

$$\Downarrow$$

$$\frac{6EI}{L}\{\theta_B\} = 135$$

$$\Downarrow$$

$$\frac{EI}{L}\theta_B = 22.5$$

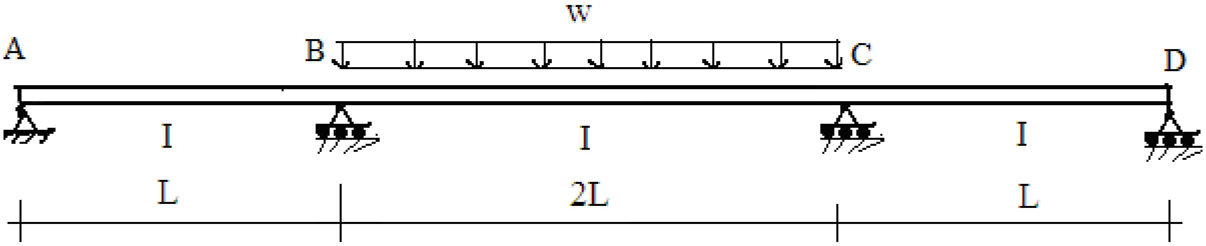
The corresponding end moments are:

M_{BA} =67.5 kN-m counterclockwise

M_{BC} =67.5 kN-m counterclockwise

Problem 10.7

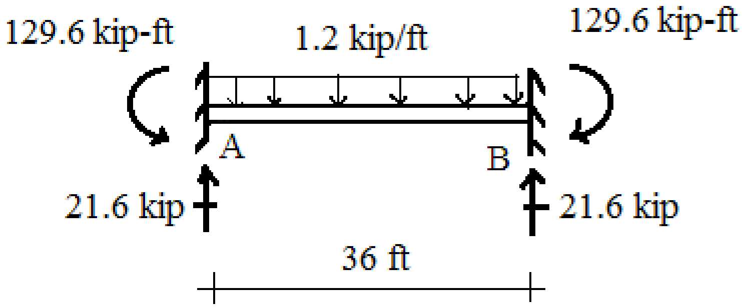
E=29,000 ksi, $I = 200\ \text{in}^4$, $L = 18\ \text{ft}$, and $w = 1.2\ \text{k/ft}$.



Because of symmetry $\theta_A = -\theta_D$ and $\theta_B = -\theta_C$

Let $k = \dfrac{EI}{L}$

$k_{\text{member BC}} = \dfrac{EI}{2L} = 0.5\ k$ $\qquad k_{\text{member AB}} = k_{\text{member CD}} = \dfrac{EI}{L} = k$



The slope-deflection equations take the form:

$M_{AB} = 0$

$M_{BA\,\text{modified}} = M_{BA\,\text{modified}} = 3k\{\theta_B\}$

$M_{BC} = k\ \{2\theta_B + \theta_C\} + M^F_{BC} = k\ \{2\theta_B - \theta_B\} + 129.6 = k\ \theta_B + 129.6$

$M_{CB} = -M_{BC}$

Enforce moment equilibrium at node B:

$$M_{BAmodified} + M_{BC} = 0$$
$$\Downarrow$$
$$4k\theta_B + 129.6 = 0$$
$$\Downarrow$$
$$k\theta_B = -32.4$$

The corresponding end moments are:

$M_{BA} = 3k\,\theta_B = -97.2 \quad \Rightarrow \quad M_{BA} = 97.2$ kip-ft clockwise

$M_{BC} = k\,\theta_B + 129.6 = 97.2$ kip-ft counterclockwise

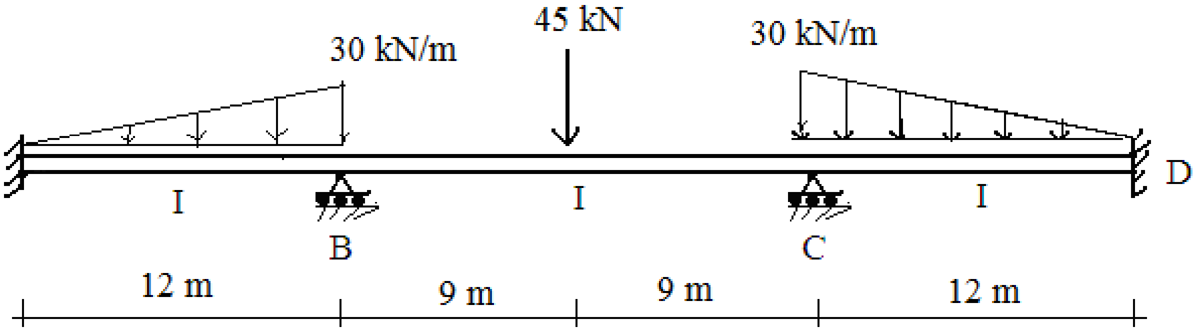
Because of symmetry

$M_{CB} = -\ M_{BC} = 97.2$ kip-ft clockwise

$M_{CD} = -M_{BA} = 97.2$ kip-ft counterclockwise

Problem 10.8

Assume E= 200 GPa and $I = 80(10)^6\ \text{mm}^4$



Let $k_{\text{member AB}} = k_{\text{member CD}} = \dfrac{EI}{12} = 1.5k \qquad k_{\text{member BC}} = \dfrac{EI}{18} = k$

$M_{AB}^F = \dfrac{30(12)^2}{30} = 144$ kN-m $\qquad M_{BA}^F = -\dfrac{30(12)^2}{20} = -216$ kN-m

$M_{BC}^F = \dfrac{45(18)}{8} = 101.25$ kN-m $\qquad M_{CB}^F = -\dfrac{45(18)}{8} = -101.25$ kN-m

$M_{CD}^F = \dfrac{30(12)^2}{20} = 216$ kN-m $\qquad M_{DC}^F = -\dfrac{30(12)^2}{30} = -144$ kN-m

Because of symmetry $\theta_B = -\theta_C$, $M_{BC} = -M_{CB}$

The slope-deflection equations take the form:

$$M_{AB} = -M_{DC} = 2(1.5k)\ \theta_B + 144$$

$$M_{BA} = -M_{CD} = 4(1.5k)\ \theta_B - 216$$

$$M_{BC} = -M_{CB} = 2k\ (2\theta_B + \theta_C) + 101.25 = 2k\ \theta_B + 101.25$$

Enforce moment equilibrium at either node B or C:

$$M_{BA} + M_{BC} = 0 \qquad 4(1.5k)\ \theta_B - 216 + 2k\ \theta_B + 101.25 = 0 \qquad k\ \theta_B = 14.34$$

The corresponding end moments are:

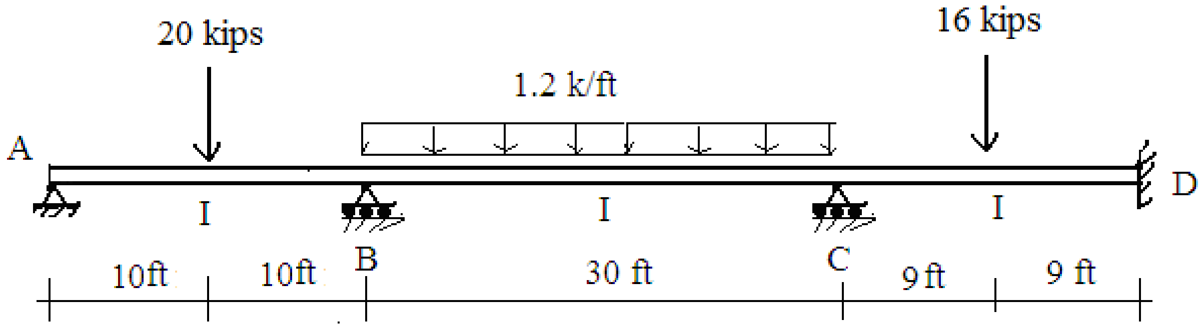
$$M_{AB} = -M_{DC} = 2(1.5k)\ \theta_B + 144 = 187\ \text{ kN-m}$$

$$M_{BA} = -M_{CD} = 4(1.5k)\ \theta_B - 216 = -129.9\ \text{ kN-m}$$

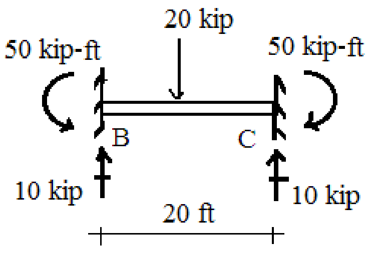
$$M_{BC} = -M_{CB} = 2k\ \theta_B + 101.25 = 129.9\ \text{ kN-m}$$

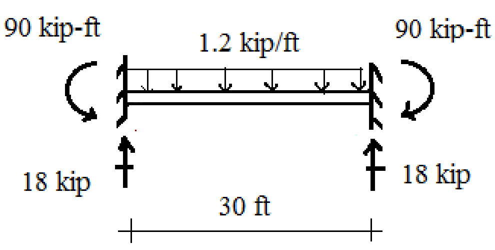
Problem 10.9

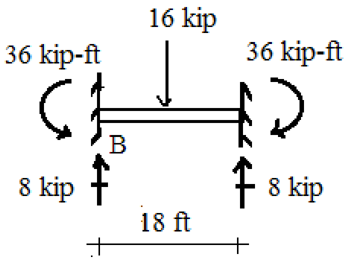
E=29,000 ksi, $I = 400\ \text{in}^4$.



Let $\quad k_{\text{member AB}} = (\frac{EI}{20}) = \frac{3}{2}k \qquad k_{\text{member BC}} = \frac{EI}{30} = k \qquad k_{\text{member CD}} = (\frac{EI}{18}) = \frac{5}{3}k$







The slope-deflection equations take the form:

$$M_{AB} = 0$$

$$M_{BA\ \text{modified}} = 3(\frac{3}{2}k)\ \{\theta_B\} - 75$$

$$M_{BC} = 2k\ \{2\theta_B + \theta_C\} + 90$$

$$M_{CB} = 2k\ \{\theta_B + 2\theta_C\} - 90$$

$$M_{CD} = 2(\frac{5}{3}k)\ \{2\theta_C\} + 36$$

$$M_{DC} = 2(\frac{5}{3}k)\ \{\theta_C\} - 36$$

Enforce moment equilibrium at nodes B and C:

$$M_{BA} + M_{BC} = 0$$

$$M_{CB} + M_{CD} = 0$$

$$\Downarrow$$

$$3(\frac{3}{2}k)\ \{\theta_B\} - 75 + 2k\ \{2\theta_B + \theta_C\} + 90 = 0$$

$$2k\ \{\theta_B + 2\theta_C\} - 90 + 2(\frac{5}{3}k)\ \{2\theta_C\} + 36 = 0$$

$$\Downarrow$$

$$8.5k\ \theta_B + 2k\ \theta_C = -15$$

$$2k\ \theta_B + 10.667k\ \theta_C = +54$$

$$\Downarrow$$

$$k\ \theta_B = -3.08 \qquad k\ \theta_C = +5.64$$

The corresponding end moments are:

$$M_{AB} = 0$$

$$M_{BA} = 3(\frac{3}{2}k)\ \{\theta_B\} - 75 = -88.9\ \text{kip-ft}$$

$$M_{BC} = 2k\ \{2\theta_B + \theta_C\} + 90 = 88.9\ \text{kip-ft}$$

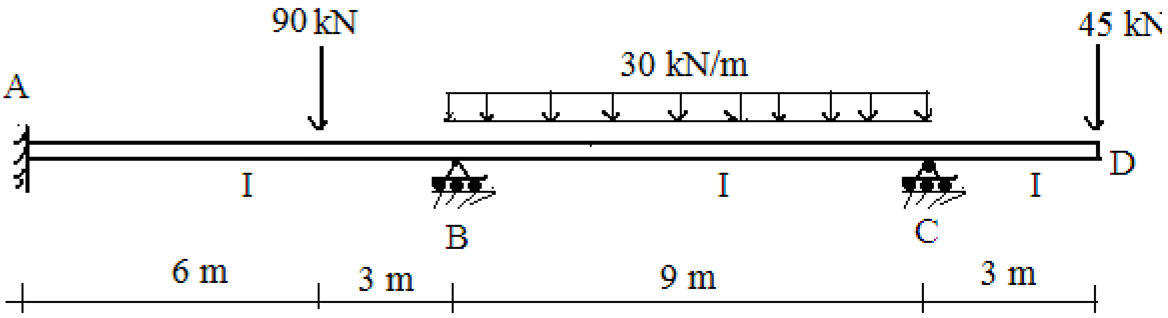
$$M_{CB} = 2k\ \{\theta_B + 2\theta_C\} - 90 = -73.6\ \text{kip-ft}$$

$$M_{CD} = 2(\frac{5}{3}k)\ \{2\theta_C\} + 36 = 73.6\ \text{kip-ft}$$

$$M_{DC} = 2(\frac{5}{3}k)\ \{\theta_C\} - 36 = -17.2\ \text{kip-ft}$$

Problem 10.10

Assume E= 200 GPa, and $I = 100(10)^6 \text{ mm}^4$.



Let $\quad k_{\text{member AB}} = k_{\text{member BC}} = \dfrac{EI}{L} = k$

$$M_{AB}^F = \frac{90(6)(3)^2}{9^2} = 60 \text{ kN-m} \qquad M_{BA}^F = -\frac{90(6)^2(3)}{9^2} = -120 \text{ kN-m}$$

$$M_{BC}^F = \frac{30(9)^2}{12} = 202.5 \text{ kN-m} \qquad M_{CB}^F = -\frac{10(20)}{8} = -202.5 \text{ kN-m}$$

The slope-deflection equations take the form:

$$M_{AB} = 2k\,\theta_B + 60$$
$$M_{BA} = 4k\,\theta_B - 120$$

$$M_{BC} = 2k\,(2\theta_B + \theta_C) + 202.5$$
$$M_{CB} = 2k\,(\theta_B + 2\theta_C) - 202.5$$

Enforce moment equilibrium at nodes B and C:

$$M_{CB} + 135 = 0 \qquad 2k\,(\theta_B + 2\theta_C) - 202.5 + 135 = 0 \quad \Rightarrow \quad k\,(\theta_B + 2\theta_C) = 33.75$$
$$M_{BA} + M_{BC} = 0 \qquad 4k\,\theta_B - 120 + 2k\,(2\theta_B + \theta_C) + 202.5 = 0 \quad \Rightarrow \quad 8k\,\theta_B + 2k\theta_C = -82.5$$

$$\therefore \qquad k\,\theta_B = -16.6 \qquad k\,\theta_C = 25.18$$

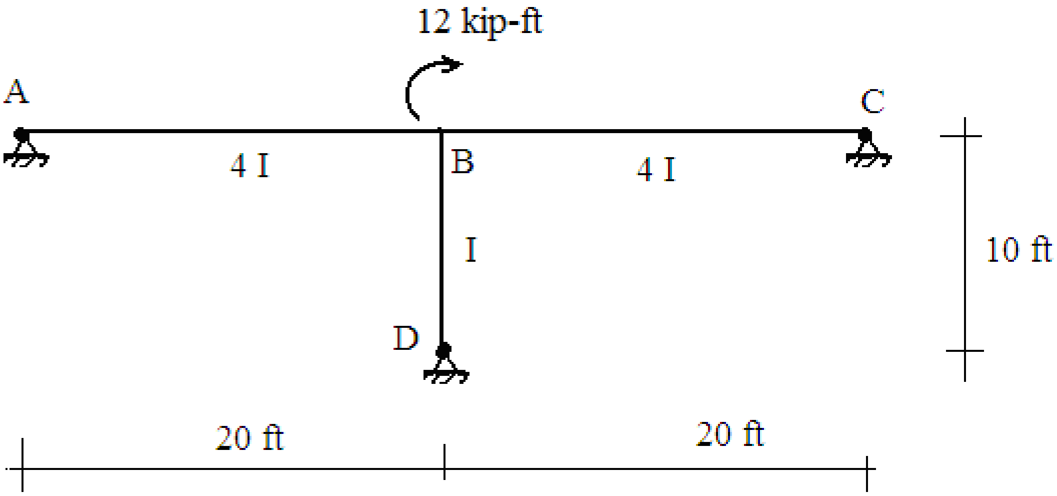
The corresponding end moments are:

$$M_{AB} = 2k\,\theta_B + 60 = 126.8 \text{ kN-m}$$
$$M_{BA} = 4k\,\theta_B - 120 = -186.4 \text{ kN-m}$$

$$M_{BC} = 2k\,(2\theta_B + \theta_C) + 202.5 = 186.4 \text{ kN-m}$$
$$M_{CB} = 2k\,(\theta_B + 2\theta_C) - 202.5 = -135 \text{ kN-m}$$

Problem 10.11

Assume $E = 29{,}000\text{ksi}$ and $I = 100 \text{ in}^4$.



Let $\quad k_{AB} = k_{BC} = \frac{E(4I)}{20}) = \frac{EI}{5} = 2k \qquad k_{BD \text{ modified}} = \frac{EI}{10} = k$

The modified slope-deflection equations take the form:

$$M_{BA \text{ modified}} = M_{BC \text{ modified}} = 3(2k)\,\theta_B$$
$$M_{BD \text{ modified}} = 3(k)\,\theta_B$$

Enforce moment equilibrium at node B:

$$M_{BA \text{ modified}} + M_{BC \text{ modified}} + M_{BD \text{ modified}} + 12 = 0$$
$$\Downarrow$$
$$3(2k)\,\theta_B + 3(2k)\,\theta_B + 3(k)\,\theta_B = -12$$
$$\Downarrow$$
$$k\,\theta_B = -0.8$$

The corresponding end moments are:

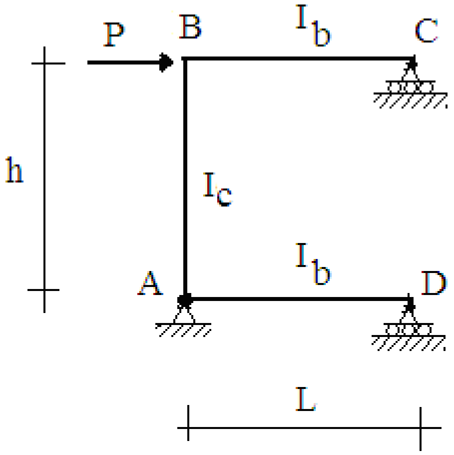
$$M_{BA \text{ modified}} = M_{BC \text{ modified}} = 3(2k)\,\theta_B = -4.8 \text{ kip-ft}$$
$$M_{BD \text{ modified}} = 3(k)\,\theta_B = -2.4 \text{ kip-ft}$$

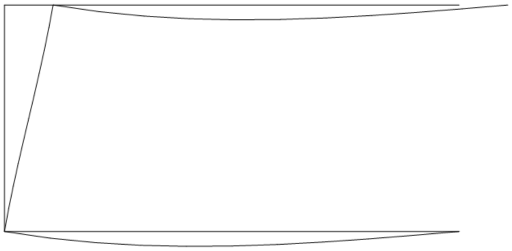
Problem 10.12

Assume $E = 200\ \text{GPa}$, $I_c = 120(10)^6\ \text{mm}^4$, $L = 8\ \text{m}$, $h = 4\ \text{m}$ and P=50 kN.

(a) $I_b = I_c$

(b) $I_b = 1.5 I_c$



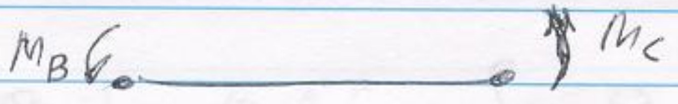


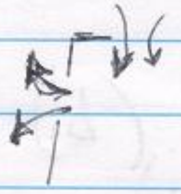
Member AD

$$M_D = \frac{2EI_B}{L}\left(2\theta_D + \theta_A\right) = 0 \Rightarrow \theta_D = \frac{-1}{2}\theta_A$$

$$M_A = \frac{2EI_B}{L}\left(2\theta_A + \theta_B\right) = \frac{2EI_B}{L}\left(\frac{3}{2}\theta_A\right) = \frac{3EI_B}{L}\theta_A$$

Member BC

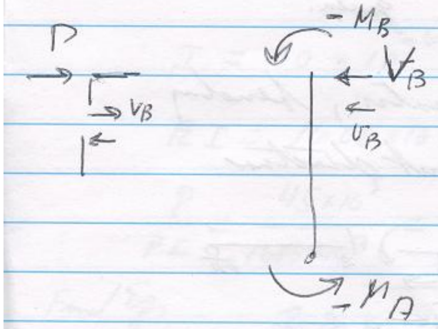




$$\theta_C = \frac{-1}{2}\theta_B$$

$$M_B = \frac{3EI_B}{L}\theta_B$$

Member AB



$$V_B = -\frac{6EI_c}{h^2}\left\{\theta_A + \theta_B - \frac{2v_B}{h}\right\}$$

$$V_B = -P$$

$$\therefore P = \frac{6EI_c}{h^2}\left\{\theta_A + \theta_B - \frac{2v_B}{h}\right\} \quad (1)$$

$$M_B = \frac{2EI_c}{h}\left\{2\theta_B + \theta_A - 3\frac{v_B}{h}\right\} = -\frac{3EI_B}{L}\theta_B \quad (2)$$

$$M_A = \frac{2EI_c}{h}\left\{2\theta_A + \theta_B - 3\frac{v_B}{h}\right\} = -\frac{3EI_B}{L}\theta_A \quad (3)$$

Solve for θ_A, θ_B, v_B. Then backsubstitute

$$\theta_A + \theta_B - \frac{2v_B}{h} = \frac{Ph^2}{6EI_c}$$

$$\cancel{2}\theta_B\left\{2 + \frac{EI_B}{L}\cdot\frac{h}{EI_c}\frac{3}{2}\right\} + \theta_A - \frac{3v_B}{h} = 0$$

$$\theta_B + \theta_A\left\{2 + \frac{3}{2}\frac{EI_B}{L}\frac{h}{EI_c}\right\} - \frac{3v_B}{h} = 0$$

Sol

$$\frac{v_B}{h} = \frac{-Ph^2}{12EI_c} + \frac{1}{2}(\theta_A + \theta_B) \qquad (a)$$

Then

$$-\frac{3v_B}{h} = +\frac{Ph^2}{4EI_c} - \frac{3}{2}(\theta_A + \theta_B)$$

and

$$\theta_B\left\{\frac{1}{2} + \frac{EI_B}{L}\frac{h}{EI_c}\frac{3}{2}\right\} - \frac{1}{2}\theta_A = -\frac{Ph^2}{4EI_c}$$

$$\theta_B\left\{-\frac{1}{2}\right\} + \theta_A\left\{\frac{1}{2} + \frac{EI_{B_1}}{L}\frac{h}{EI_c}\frac{3}{2}\right\} = -\frac{Ph^2}{4EI_c} \qquad (b)$$

Solve numerically using MATCAD and the dimensions given.

(a) $I_b = I_c = 120(10)^6\ \text{mm}^4 \quad I_c = 120(10)^6\ \text{mm}^4$

$\theta_A =$

$\theta_B =$

$v_B =$

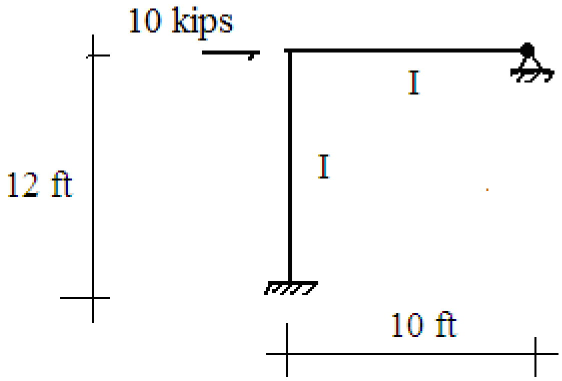
$M_{AD} =$

$M_{AB} =$

(b) $I_c = 120(10)^6\ \text{mm}^4$**,** $I_b = 1.5I_c = 180(10)^6\ \text{mm}^4$

Problem 10.13

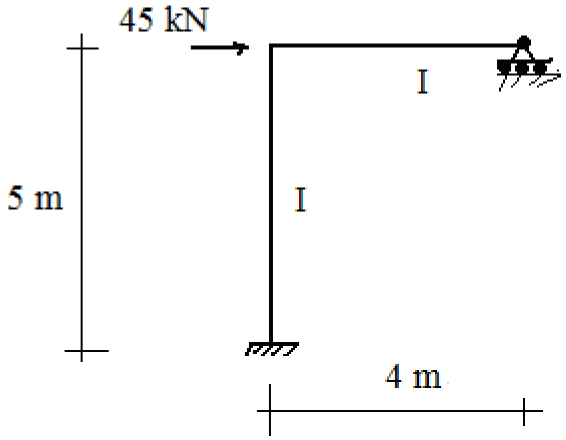
Assume $E = 29{,}000\text{ksi}$ and $I = 200\ \text{in}^4$.

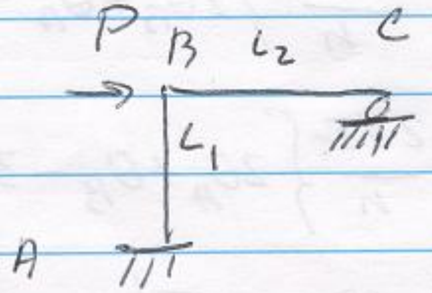


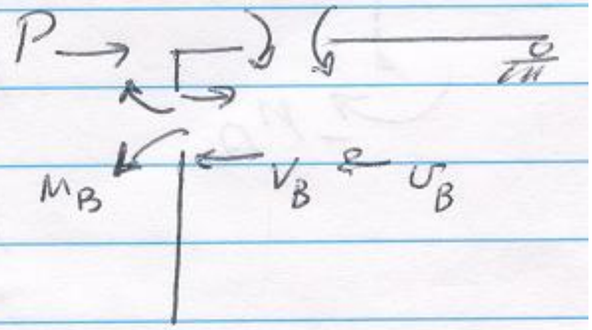
For no axial deformation, bending moment is zero throughout the structure.

Problem 10.14

Assume $E = 200\ \text{GPa}$, and $I = 80(10)^6\ \text{mm}^4$.





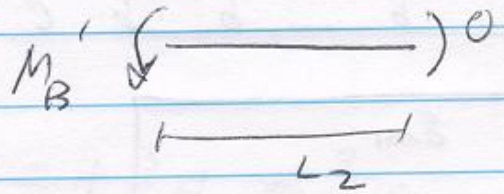


Member AB

$$V_B = -P = -\frac{6EI}{L_1^2}(\cancel{2}\theta_B) + \frac{12EI}{L_1^3} v_B$$

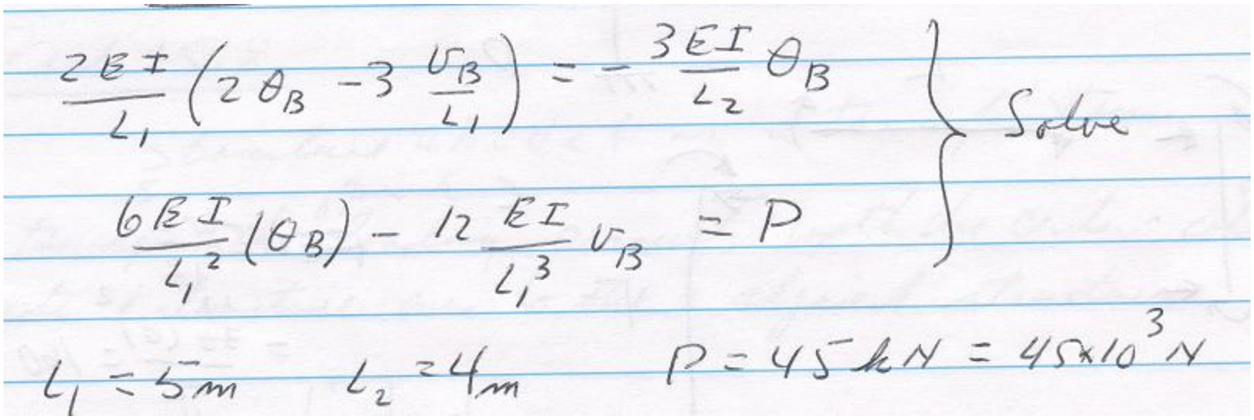
$$M_B = \frac{2EI}{L_1}\left(2\theta_B - 3\frac{v_B}{L_1}\right)$$

Member BC



$$\theta_c = -\frac{1}{2}\theta_B$$

$$M_B' = \frac{3EI\theta_B}{L_2} = -M_B$$

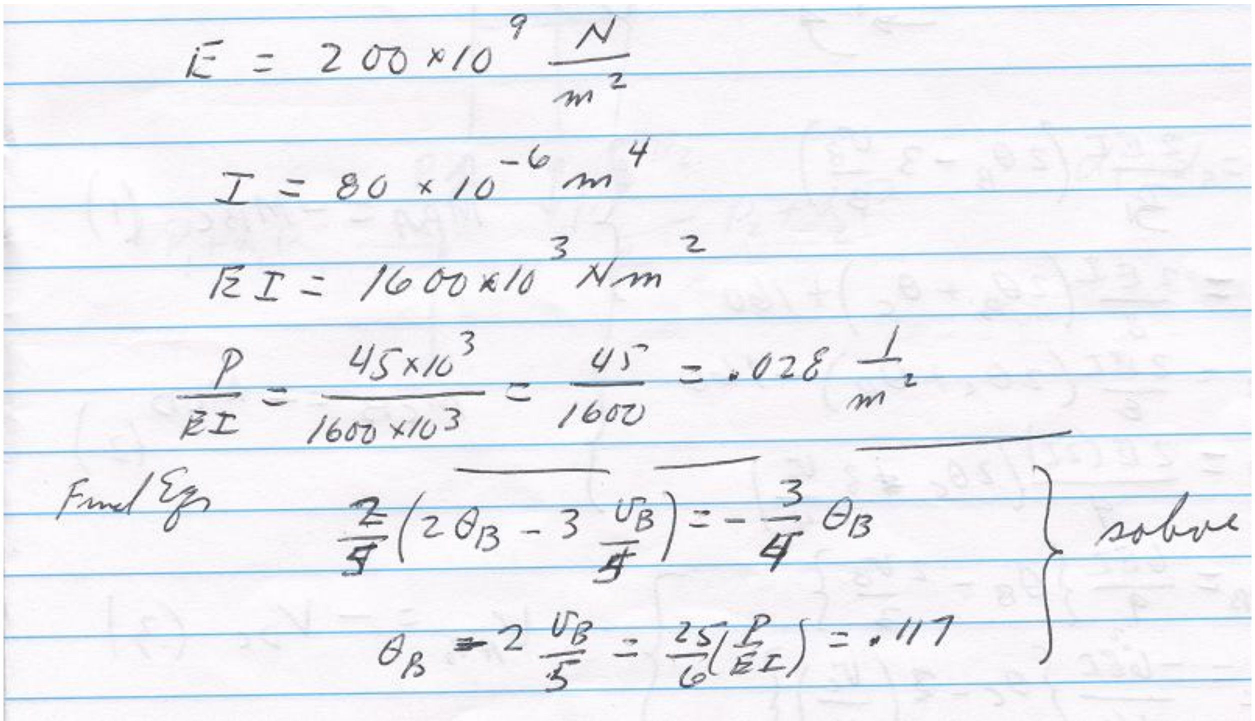


$$\frac{2EI}{L_1}\left(2\theta_B - 3\frac{v_B}{L_1}\right) = -\frac{3EI}{L_2}\theta_B$$

$$\frac{6EI}{L_1^2}(\theta_B) - 12\frac{EI}{L_1^3}v_B = P$$

Solve

$L_1 = 5m \quad L_2 = 4m \quad P = 45kN = 45\times10^3 N$



$E = 200\times10^9 \frac{N}{m^2}$

$I = 80\times10^{-6} m^4$

$EI = 1600\times10^3 Nm^2$

$\frac{P}{EI} = \frac{45\times10^3}{1600\times10^3} = \frac{45}{1600} = .028 \frac{1}{m^2}$

Final Eqs

$$\frac{2}{5}\left(2\theta_B - 3\frac{v_B}{5}\right) = -\frac{3}{4}\theta_B$$

$$\theta_B - 2\frac{v_B}{5} = \frac{25}{6}\left(\frac{P}{EI}\right) = .117$$

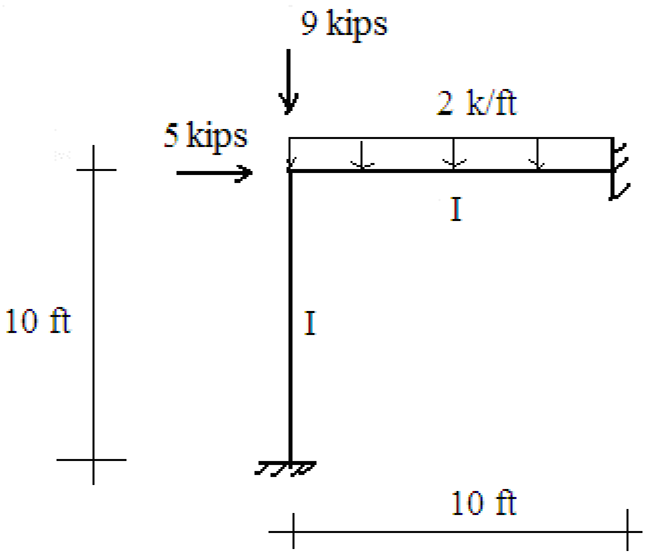
solve

Problem 10.15

$\mathrm{I} = 600 \ \mathrm{in}^4$

$\mathrm{A} = 6 \ \mathrm{in}^2$

$\mathrm{E} = 29,000 \ \mathrm{k/in}^2$

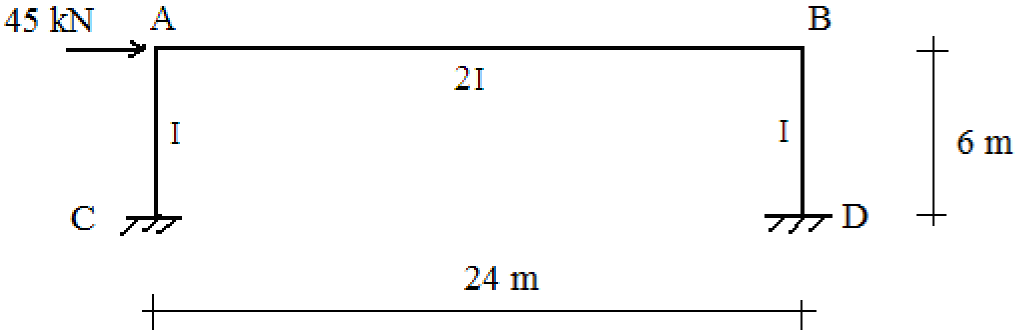


Problem 10.16

Assume $E = 200$ GPa , and $I = 120(10)^6$ mm^4.

Neglecting axial deformation.

$H_C = H_D = 22.5,\ \theta_A = \theta_C$



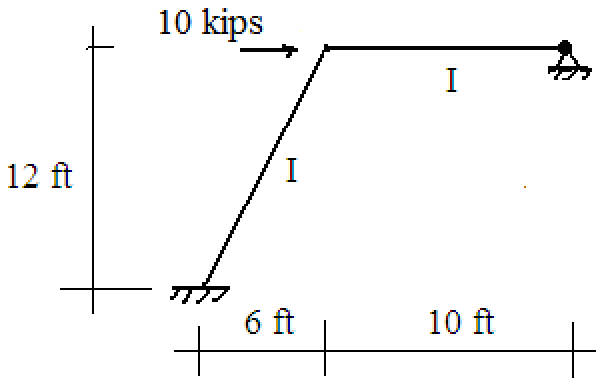
$$k_{AC} = \frac{EI}{6} = 2k \qquad k_{BC} = \frac{E(2I)}{24} = k$$

$$M_{CA} = 4k\,(\theta_A + 3\rho)$$
$$M_{AC} = 4k\,(2\theta_A + 3\rho)$$
$$M_{AB} = 2k\,(2\theta_A + \theta_C) = 6k\,\theta_A$$

$$M_{AB} = M_{BA} = -50.6 \text{ kN-m}$$
$$M_{AC} = M_{BD} = 50.6 \text{ kN-m}$$
$$M_{CA} = M_{DB} = 84.4 \text{ kN-m}$$

Problem 10.17

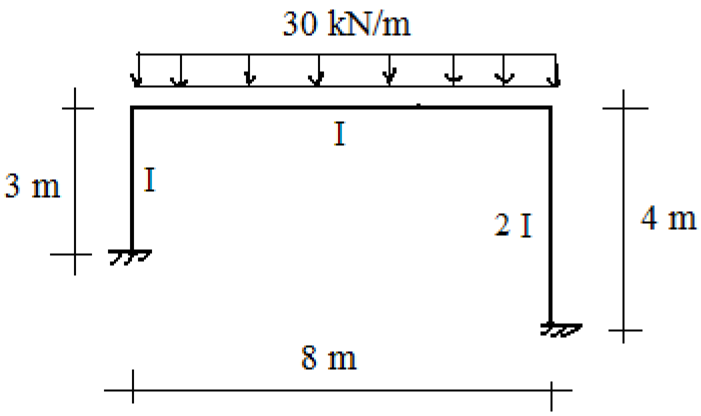
Assume $E = 29{,}000\text{ksi}$ and $I = 200\ \text{in}^4$.

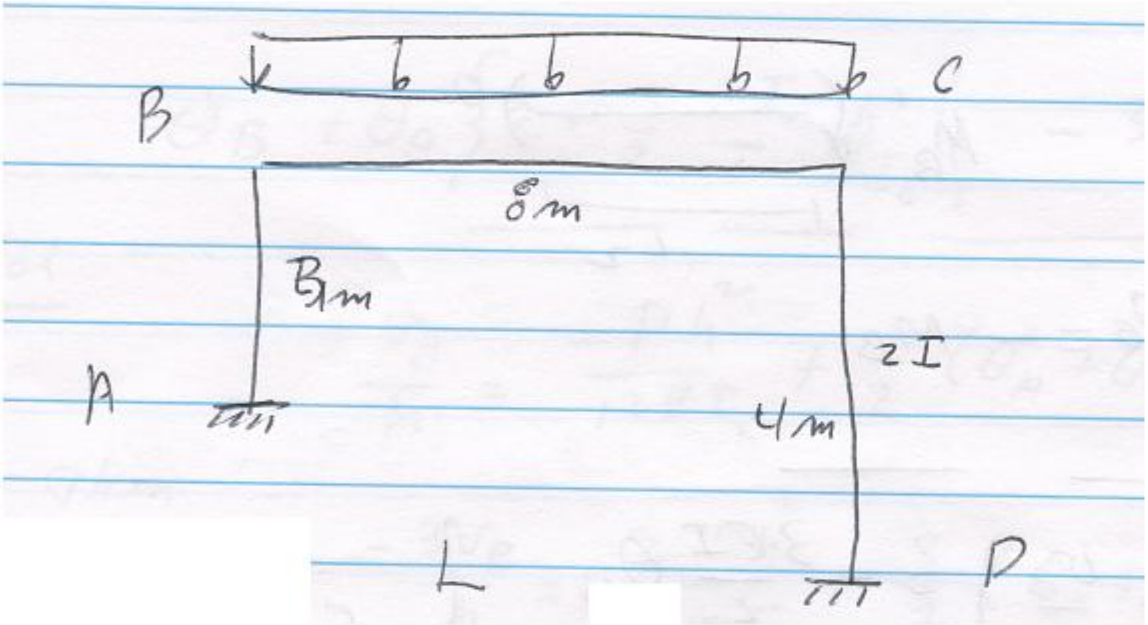


For no axial deformation, bending moment is zero throughout the structure.

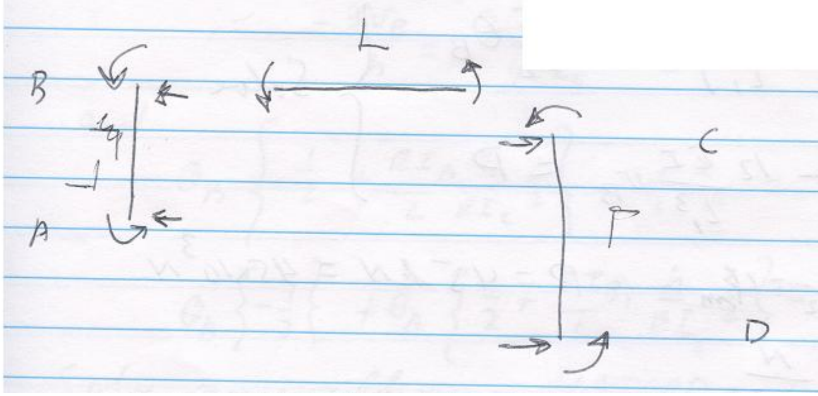
Problem 10.18

Assume $E = 200\ \text{GPa}$, and $I = 80(10)^6\ \text{mm}^4$.





Note $V_C = V_B$



$$FEM = \frac{-wL^2}{12}$$

$$= \frac{\overset{10}{\cancel{30}}(8)^2}{\underset{4}{\cancel{12}}} = 160\ kNm$$

$$M_{BA} = \frac{2EI}{3}\left(2\theta_B - 3\frac{v_B}{3}\right)$$

$$M_{BC} = \frac{2EI}{8}\left(2\theta_B + \theta_C\right) + 160$$

$$M_{BA} = -M_{BC} \quad (1)$$

$$M_{CB} = \frac{2EI}{8}\left(2\theta_C + \theta_B\right) - 160$$

$$M_{CD} = \frac{2E(2I)}{4}\left(2\theta_C - 3\frac{v_C}{4}\right)$$

$$M_{CB} = -M_{CD} \quad (2)$$

$$V_{AB} = \frac{6EI}{9}\left\{\theta_B - 2\frac{v_B}{3}\right\}$$

$$V_{DC} = \frac{-6EI}{16}\left\{\theta_C - 2\left(\frac{v_C}{4}\right)\right\}$$

$$V_{AB} = -V_{DC} \quad (3)$$

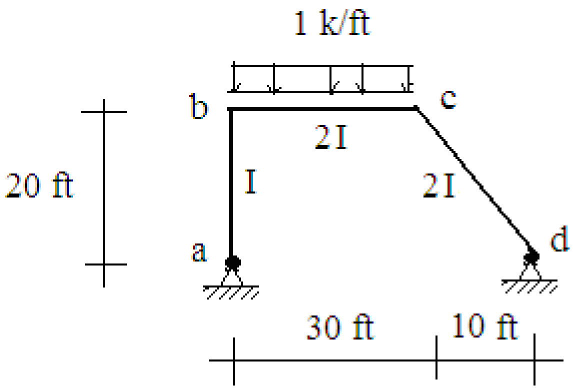
Eq (1) $\quad \frac{4}{3}\theta_B - \frac{2}{3}v_B = \frac{1}{2}\theta_B + \frac{1}{4}\theta_C + 160$

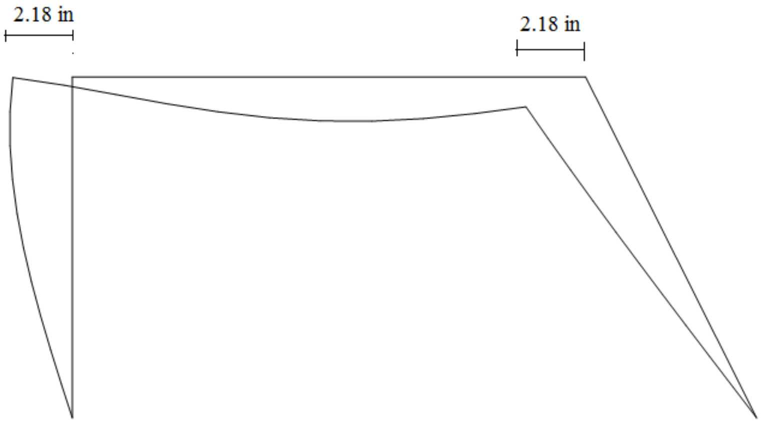
Eq (2) $\quad \frac{1}{2}\theta_C + \frac{1}{4}\theta_B = -2\theta_C + \frac{3}{4}v_C$

Eq (3) $\quad \frac{2}{3}\theta_B - \frac{4}{9}v_B = +\frac{3}{4}\theta_C - \frac{3}{8}v_C$

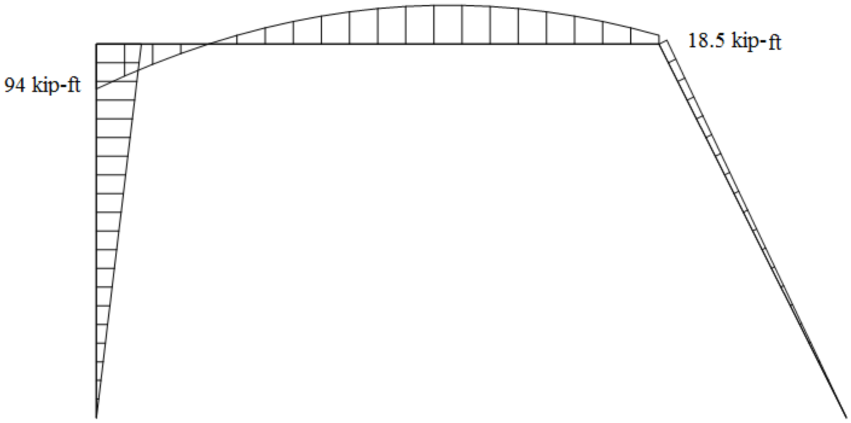
Problem 10.19

Assume $E = 29{,}000\text{ksi}$ and $I = 200 \text{ in}^4$.





Deflection profile

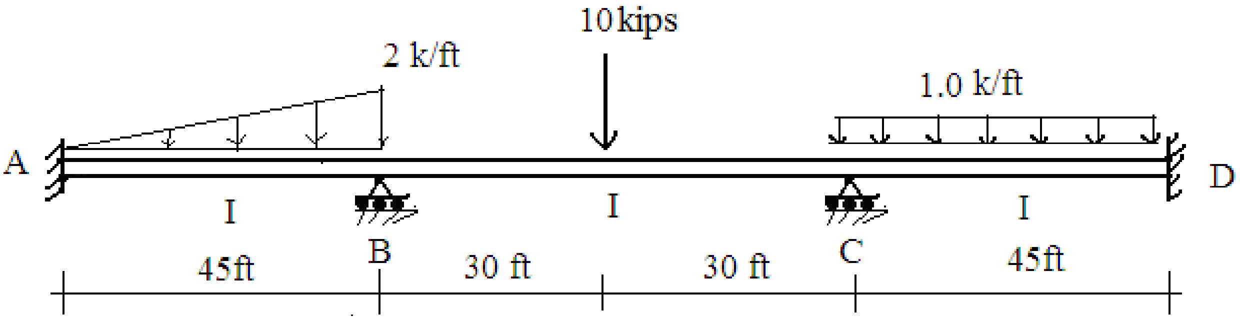


Moment Diagram

For the following beams and frames defined in Problems 10.20- 10.34, determine the member end moments using moment distribution.

Problem 10.20

E = 29,000 ksi, I = 300 in^4



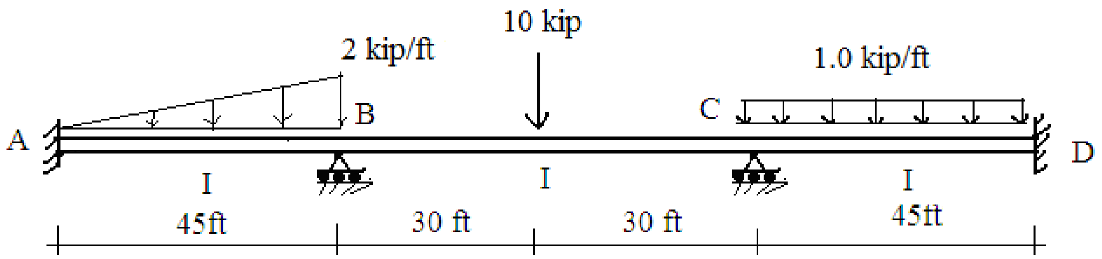
$$M^F{}_{AB} = \frac{2(45)^2}{30} = 135 \text{ kip-ft} \qquad M^F{}_{BC} = \frac{10(60)}{8} = 75 \text{ kip-ft} \qquad M^F{}_{CD} = \frac{1.0(45)^2}{12} = 168.75 \text{ kip-ft}$$

$$M^F{}_{BA} = -\frac{2(45)^2}{20} = -\ 202.5 \text{ kip-ft} \qquad M^F{}_{CB} = -\ 75 \text{ kip-ft} \qquad M^F{}_{DC} = -\,168.75 \text{ kip-ft}$$

$$DF_{DC} = DF_{AB} = 0$$

$$\text{At joints B or C}\left\{\begin{array}{l} \sum \frac{I}{L} \ = \ (\frac{I}{45}) + (\frac{I}{60}) = \frac{7\ I}{180} \\ DF_{BA} = DF_{CD} = \dfrac{\frac{I}{45}}{\frac{7\ I}{180}} = \frac{4}{7} \\ DF_{CB} = DF_{BC} = 1\text{-}\ \frac{4}{7} = \frac{3}{7} \end{array}\right.$$

Case(a):

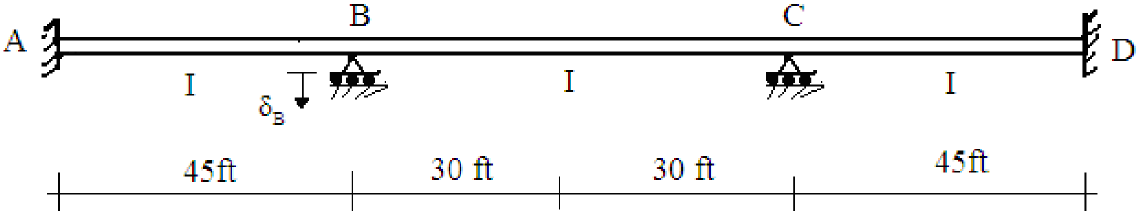


DF's			4/7	3/7		3/7	4/7		
FEM's	135		-202.5	75		-75	168.75		-168.75
				-20.08	←	-40.18	-53.42	→	-26.71
	42.16	←	84.33	63.25	→	31.62			
				-6.77	←	-13.55	-18	→	-9
	1.93	←	3.86	2.9	→	1.45			
				-.31	←	-.62	-.82	→	-.41
			.18	.13					
$\sum M$	179.1		- 114.1	114.1		-96.3	96.4		-204.9

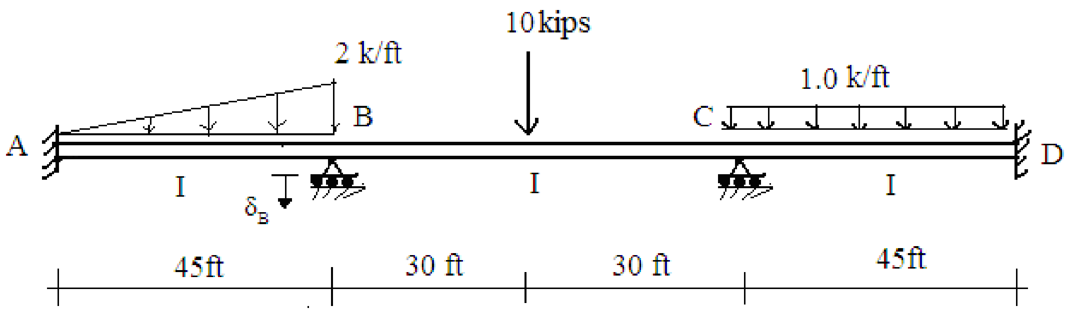
Case (b):

$$M^F{}_{AB\ set.} = M^F{}_{BA\ set.} = \frac{6EI\delta_B}{L^2} = \frac{6(29000)(300)(.5)}{(45)^2(12)^3} = 7.45\ \text{kip-ft}$$

$$M^F{}_{BC\ set.} = M^F{}_{CB\ set.} == \frac{6(29000)(300)(.5)}{(60)^2(12)^3} = 4.19\ \text{kip-ft}$$

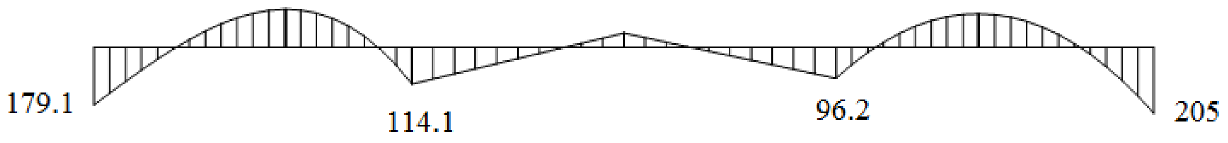


DF's		4/7	3/7	3/7	4/7	
FEM's	7.45	7.45	-4.19	-4.19		
			.9	1.8	2.39	1.19
	-1.19	-2.38	-1.78	-.89		
			.19	.38	.5	.25
	-.05	-.11	-.08			
$\sum M$	6.2	4.96	-4.93	-2.9	2.89	1.44

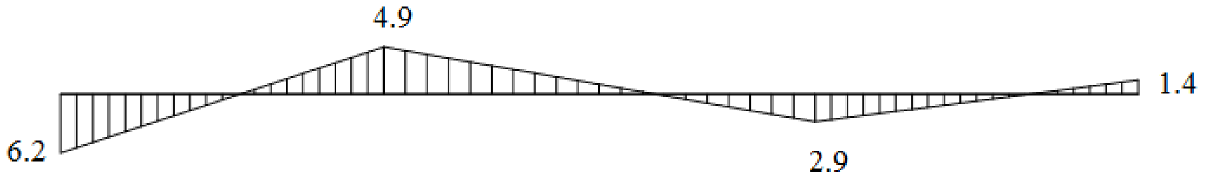


set. ∑M	6.2	4.96	-4.93	-2.9	2.89	1.44
load ∑M	179.1	- 114.1	114.1	-96.3	96.4	-204.9
Total ∑M	185.3	-109.1	109.1	-99.2	99.3	-203.5

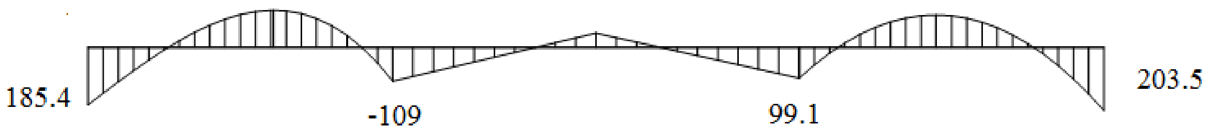
Case (c):



Moment diagram-loading kip-ft



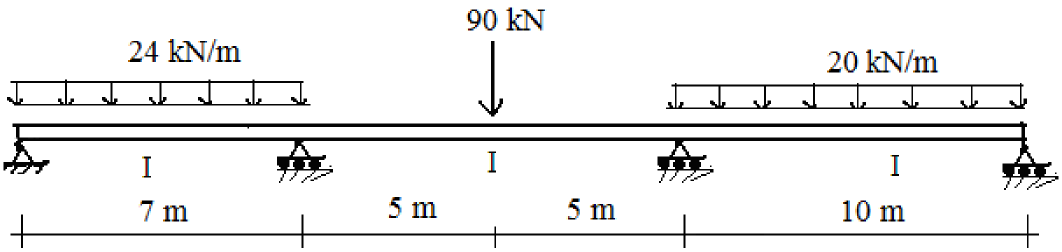
Moment diagram –settlement kip-ft



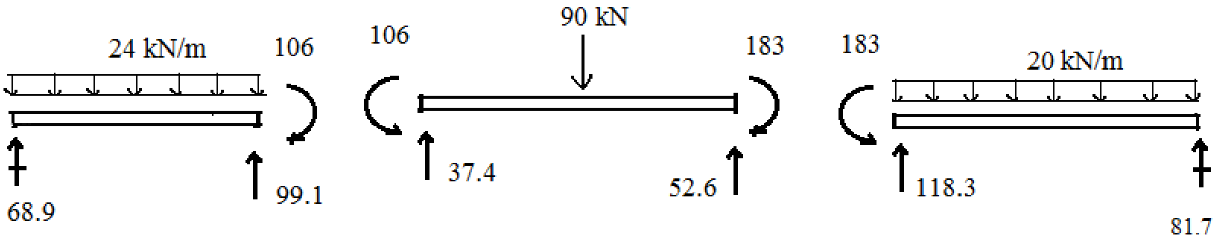
Moment diagram -(loading + settlement) kip-ft

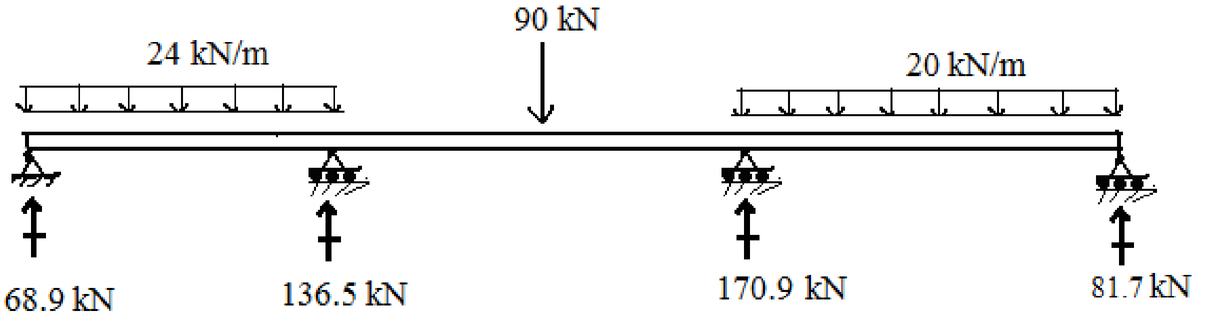
Problem 10.21

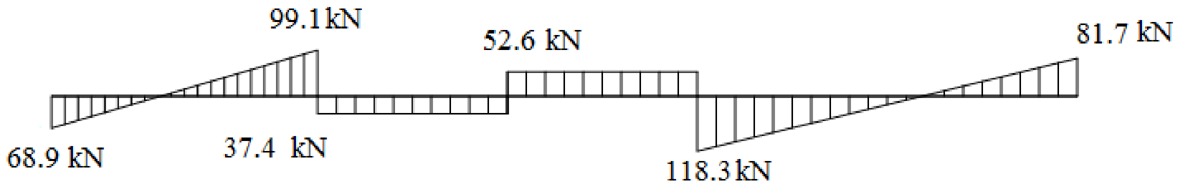
Check your results with Computer analysis. Assume $E = 200\ \text{GPa}$, and $I = 75(10)^6\ \text{mm}^4$.



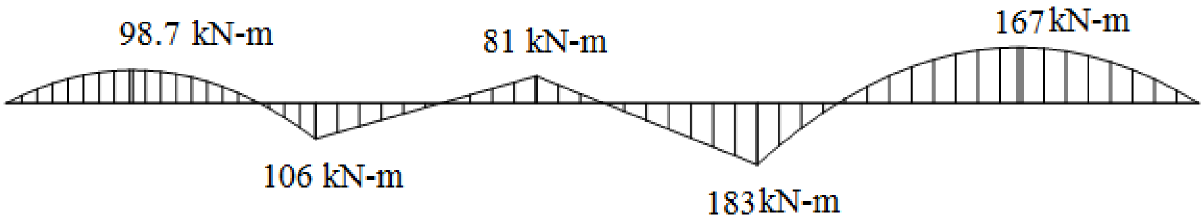
DF's	0	.52	.48		.57	.43	0
FEM's		-147	112.5		-112.5	250	
		17.94	16.56	→	8.28		
			-41.5	←	-83	-62.68	
		21.58	19.92	→	9.96		
			-2.8	←	-5.7	-4.3	
		1.45	1.34				
$\sum M$	0	-106	106		-183	183	0







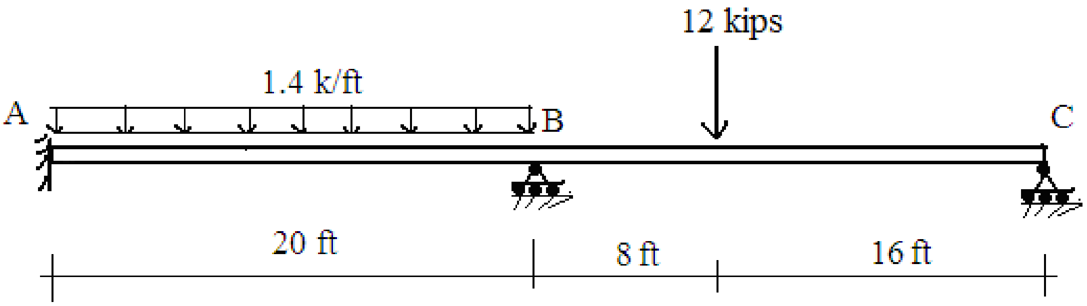
Shear diagram



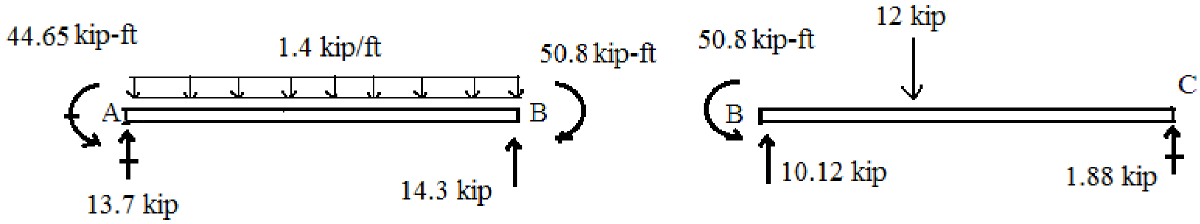
Moment diagram

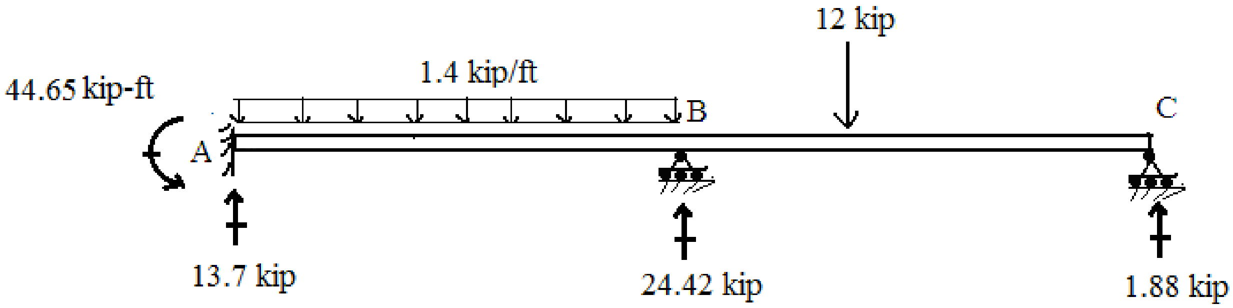
Problem 10.22
Assume EI constant.

(a)

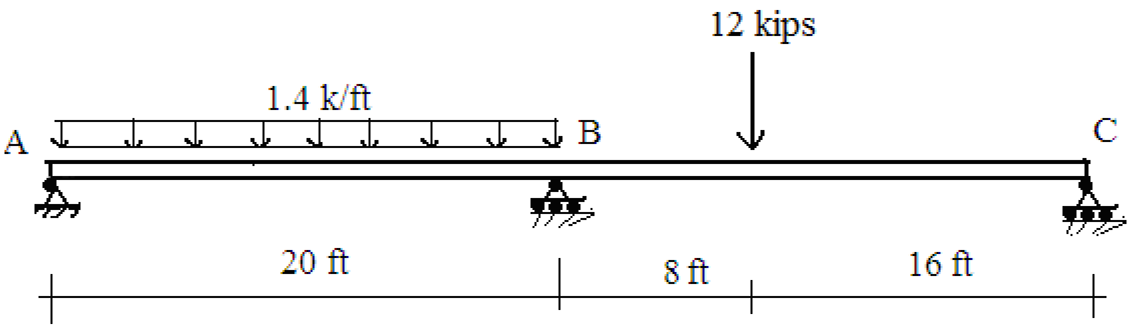


DF's	1	.615	.385	0
FEM's	+46.7	-46.7	53.3	
	- 2.05	← - 4.1	- 2.5	
∑M	44.65	-50.8	50.8	0

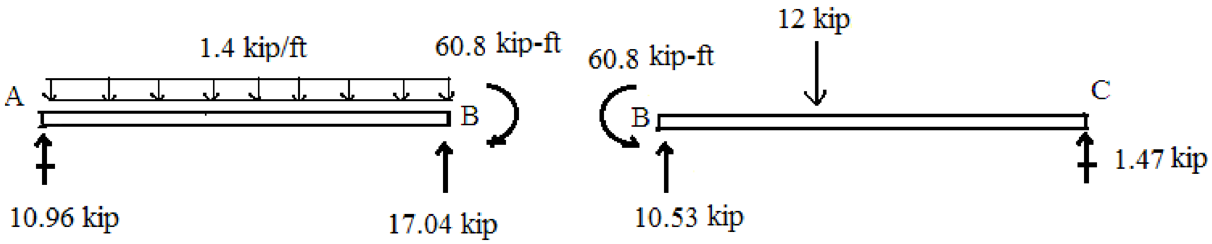


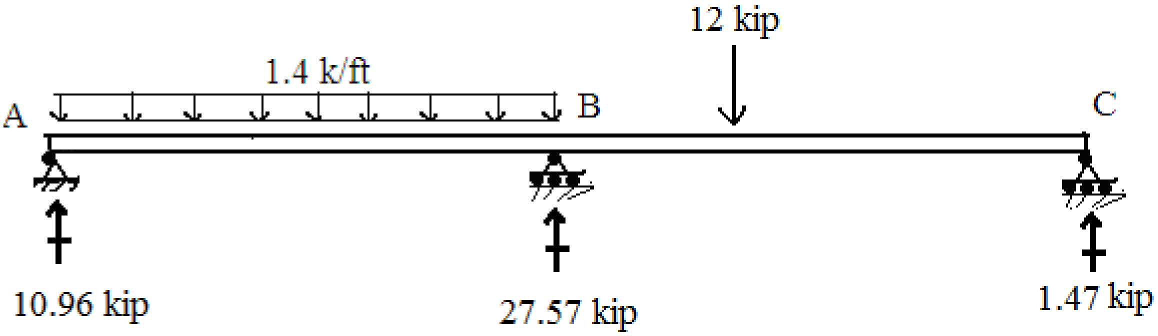


(b)



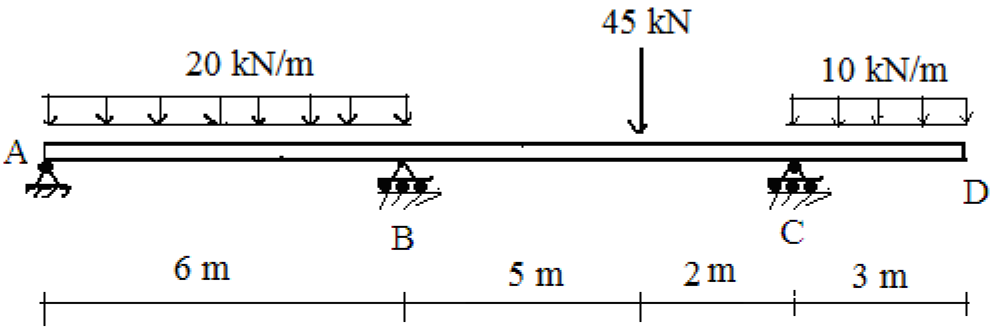
DF's	0	.55	.45	0
FEM's		-70	53.3	
		9.18	7.5	
$\sum M$	0	-60.8	60.8	0





Problem 10.23

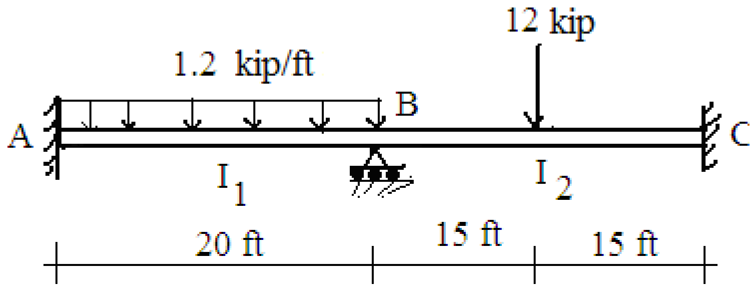
Determine the bending moment distribution. Assume EI is constant.



DF's		.54	.46		
FEM's	0	-90	41.32	0	
				-45	45
			-22.5		
		38.43	32.74		
$\sum M$	0	-51.57	51.56	-45	45

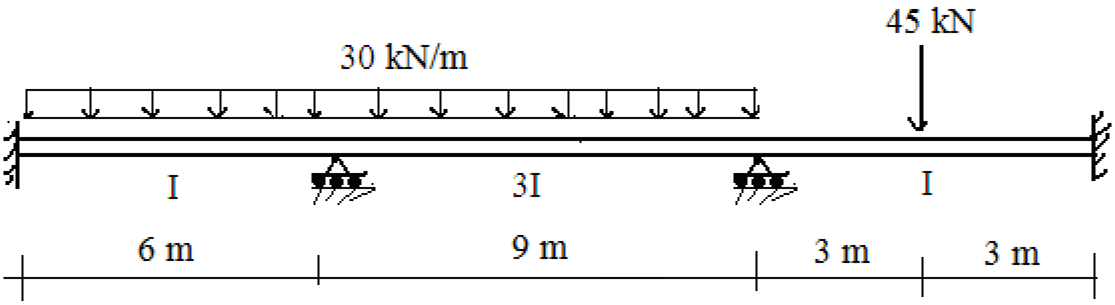
Problem 10.24

Determine the bending moment distribution. Assume $I_1 = 1.4 I_2$.

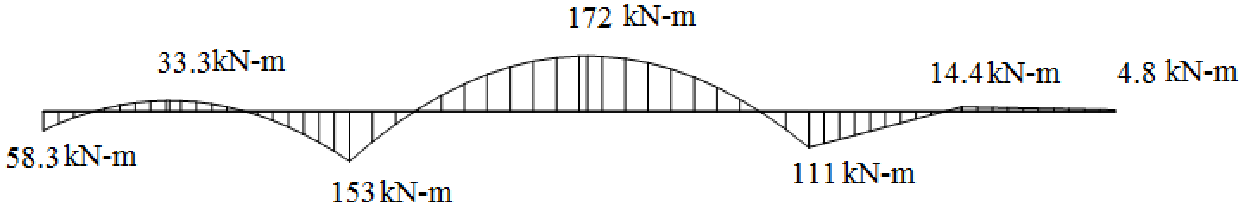


DF's	0	0.68	0.32	0
FEM's	40	-40	45	-45
	-1.7 ←	-3.4	-1.6	→ -.8
$\sum M$	38.3	-43.4	43.4	- 45.8

Problem 10.25
Determine the bending moment distribution.

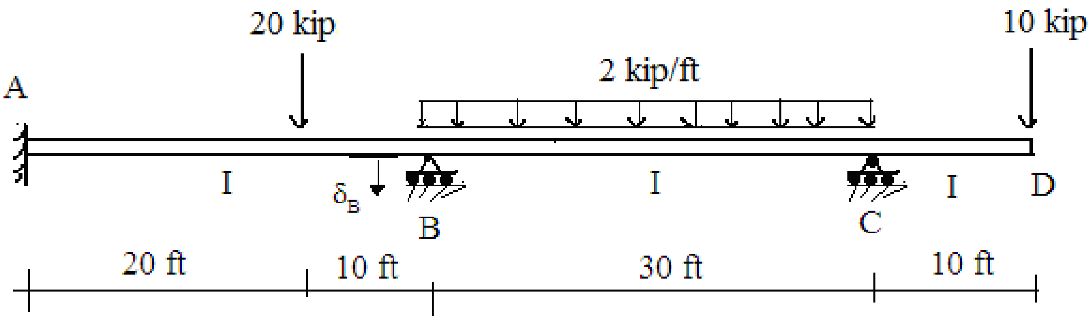


DF's		1/3	2/3	2/3	1/3	
FEM's	90	-90	202.5	-202.5	33.75	-33.75
	-18.75 ←	-37.5	-75 →	-37.5		
			68.75 ←	137.5	68.75 →	34.375
	-11.45 ←	-22.9	-45.8 →	-22.9		
			7.63 ←	15.27	7.63 →	3.82
	-1.27 ←	-2.54	-5.1 →	-2.54		
			.84 ←	1.68	.84 →	.4
		-.3	-.56			
$\sum M$	58.3	-153	153	-111	111	4.8



Problem 10.26

Solve for the bending moments. $\delta_B = .4$ in ↓, E=29,000 ksi and $I = 240\ \text{in}^4$.

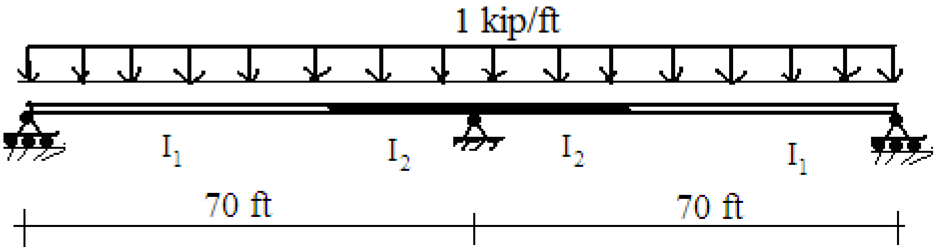


DF's		.57	.43	1	0
FEM's	44.4	-88.8	225	-100	+100
	10.74	10.74	- 5.37		
			-50		
	-26	-52.2	-39.37		
$\sum$M	29.1	-130.2	130.2	-100	+100

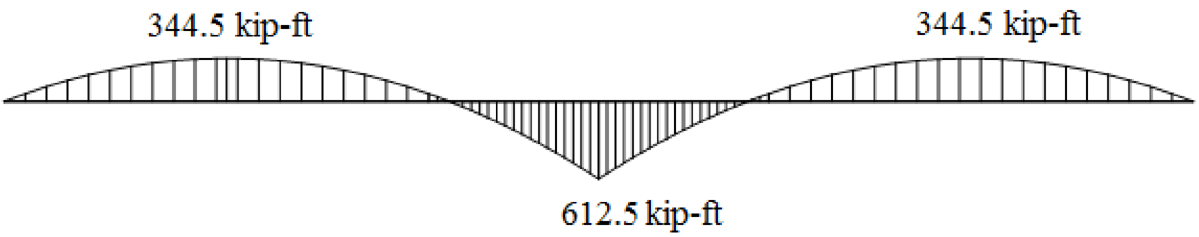
Problem 10.27

Determine the bending moment distribution and the deflected shape. E=29,000 ksi

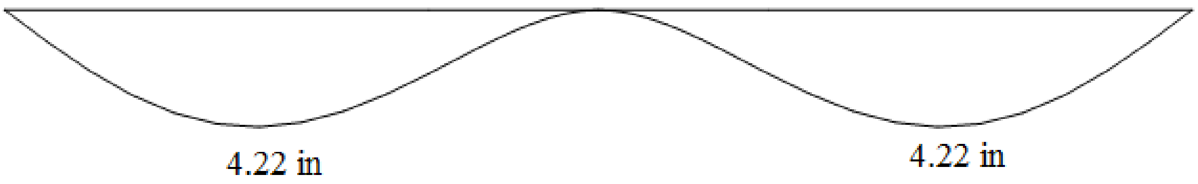
(a) Take $I_1 = I_2 = 1000 \text{ in}^4$



DF's	0	0.5	0.5	0
FEM's		-612.5	+612.5	
$\sum M$	0	-612.5	+612.5	0

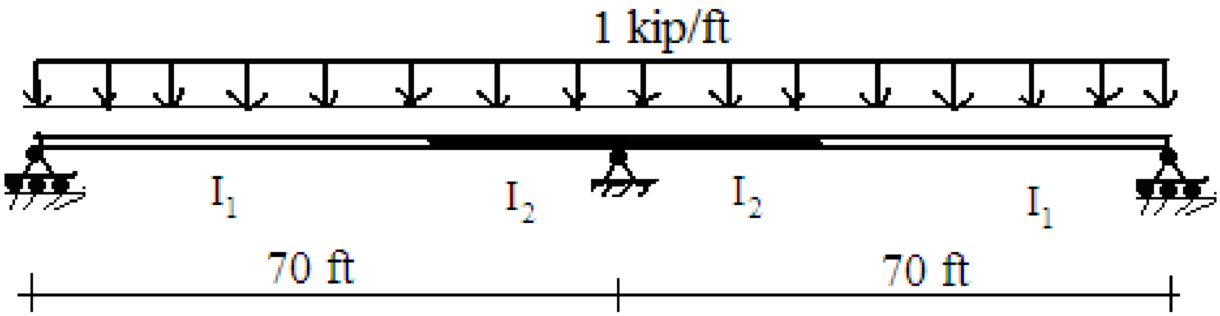


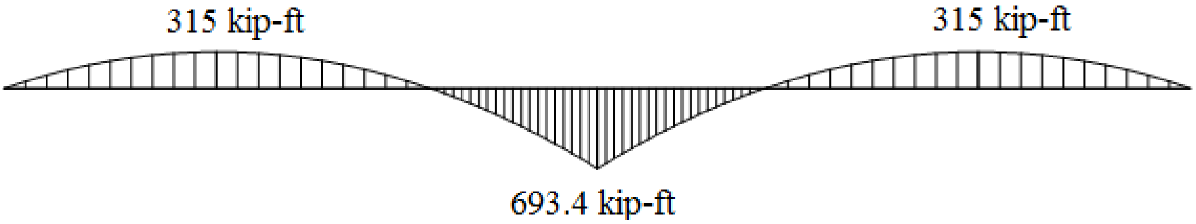
Moment diagram



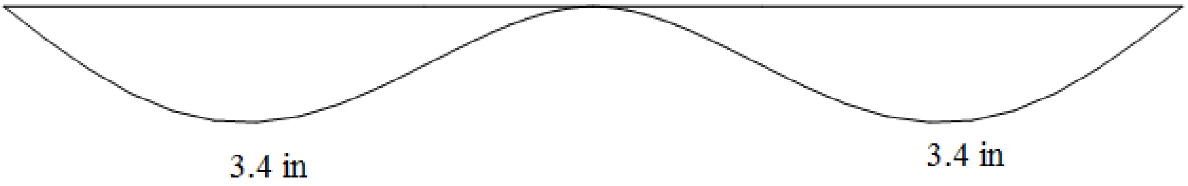
Deflected shape

(b) Take $I_2 = 1.5\ I_1$. Use Computer analysis.





Moment diagram

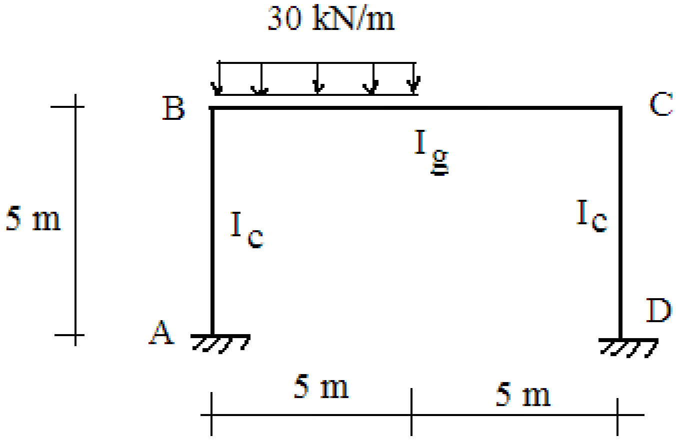


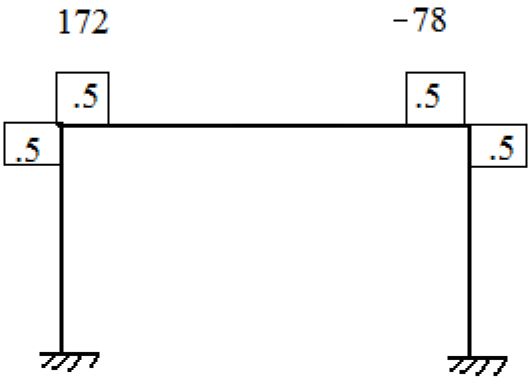
Deflected shape

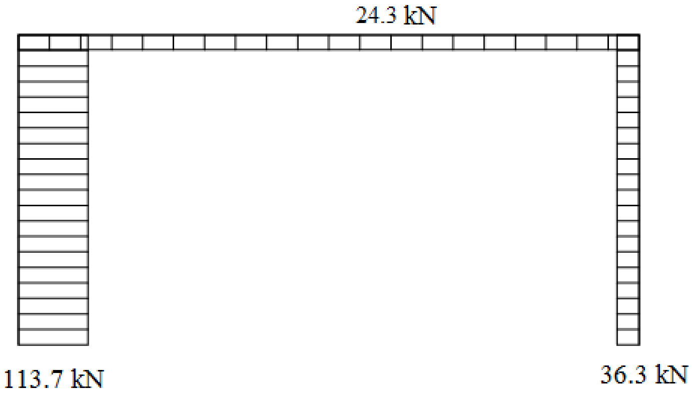
Case (a) has larger moment and deflection compare to case(b).

Problem 10.28

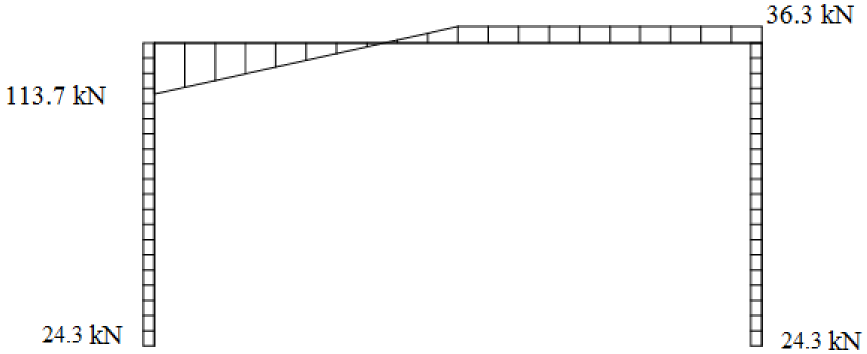
Determine the axial, shear, and bending moment distributions. Take $I_g = 2I_c$



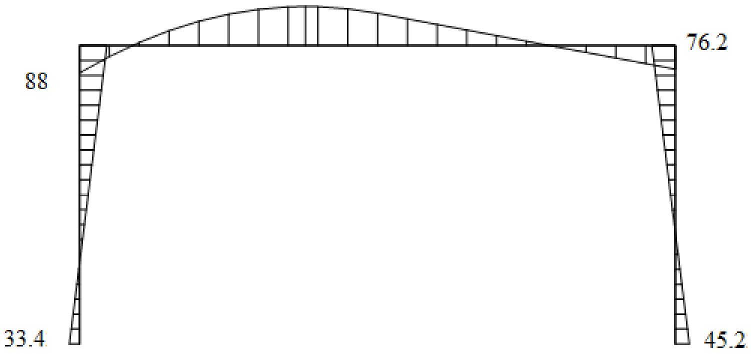




Axial Force



Shear diagram



Moment Diagram (kN-m)

Problem 10.29

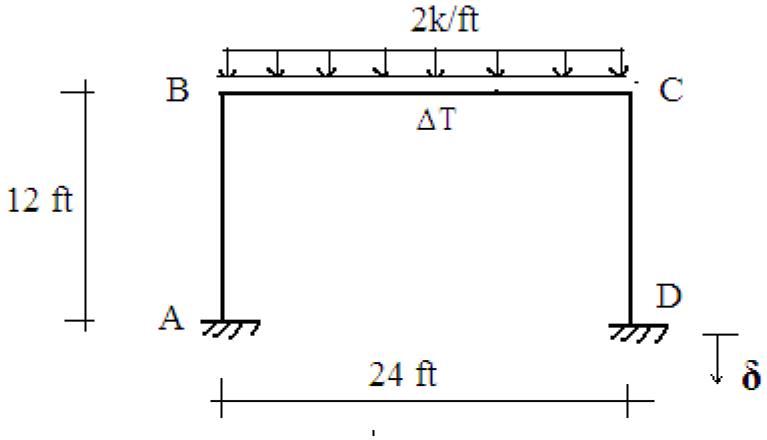
Determine the member forces and the reactions.

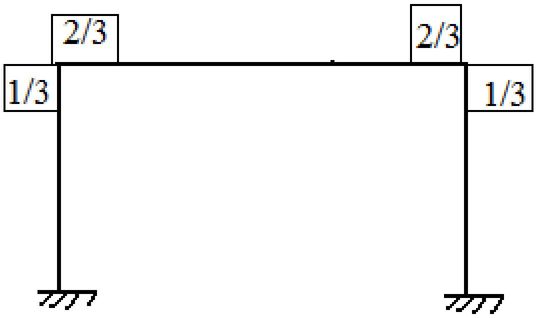
a) Consider only the uniform load shown

b) Consider only the support settlement of joint D (δ_D = .5 in↓)

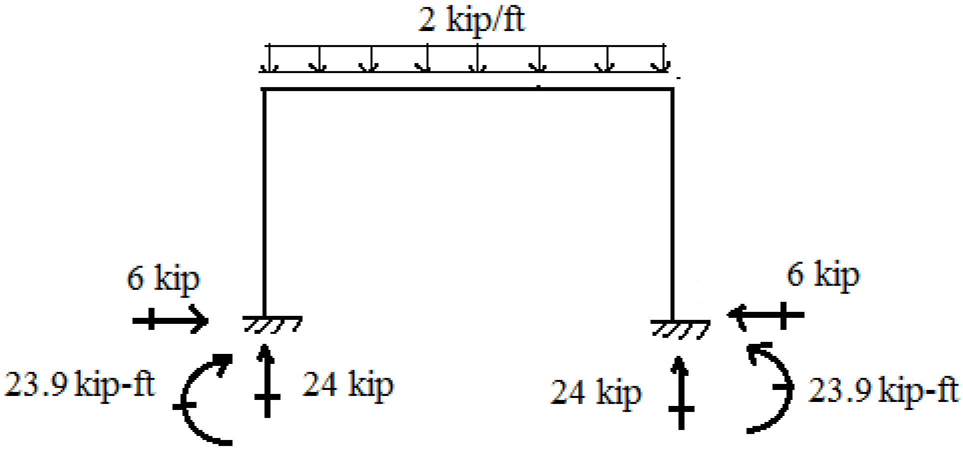
c) Consider only the temperature increase of $\Delta T = 80^{/0\ F}$ for member BC.

$E = 29{,}000$ ksi, $I_{AB} = I_{CD} = 100\ in^4$, $I_{BC} = 400\ in^4$, $\alpha = 6.5x10^{-6}/\ ^0F$

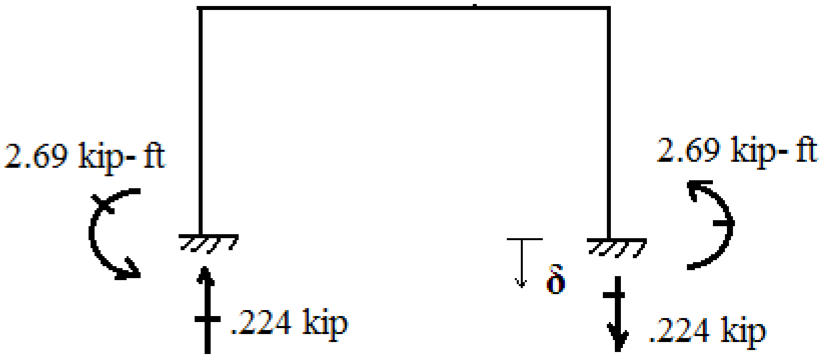



2/3
2/3
1/3
1/3

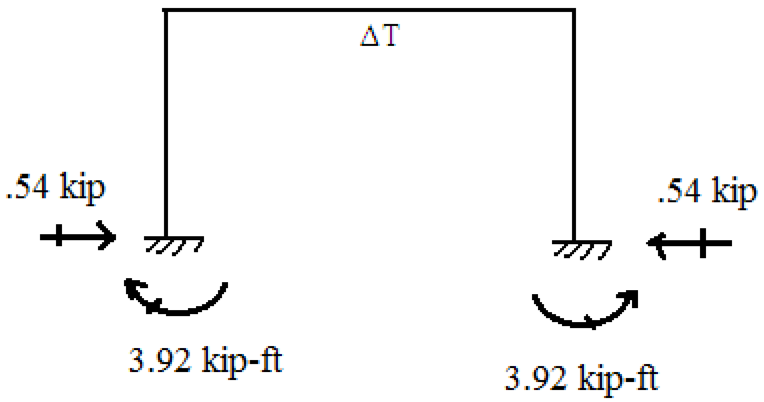
(a)


2 kip/ft
6 kip
6 kip
23.9 kip-ft
24 kip
24 kip
23.9 kip-ft

(b)

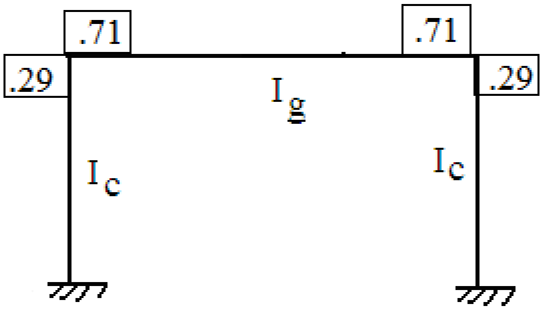

2.69 kip- ft
2.69 kip- ft
δ
.224 kip
.224 kip

(c)

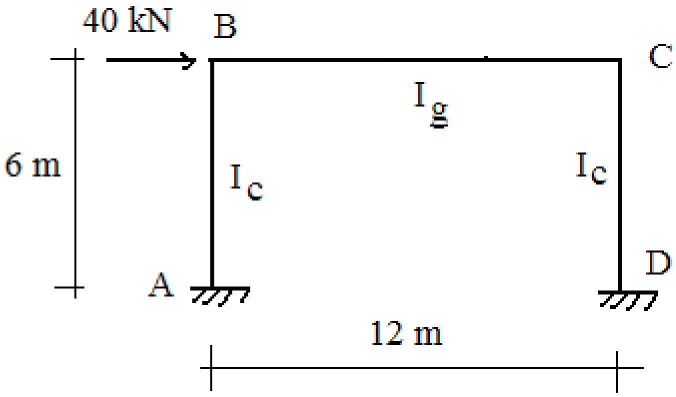


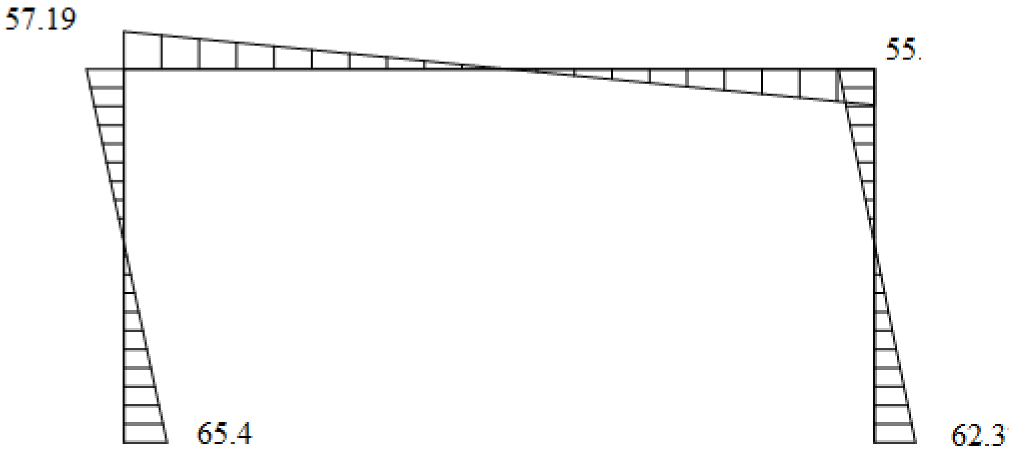
Problem 10.30

Determine the bending moment distribution for the following loadings. Take $I_g = 5I_c$.



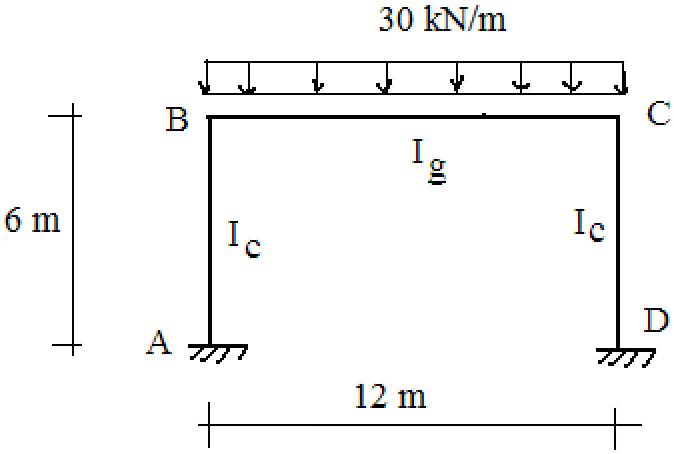
(a)

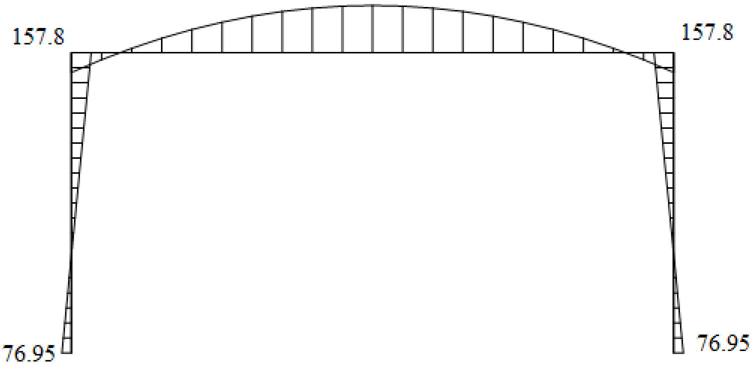




Moment diagram (kN-m)

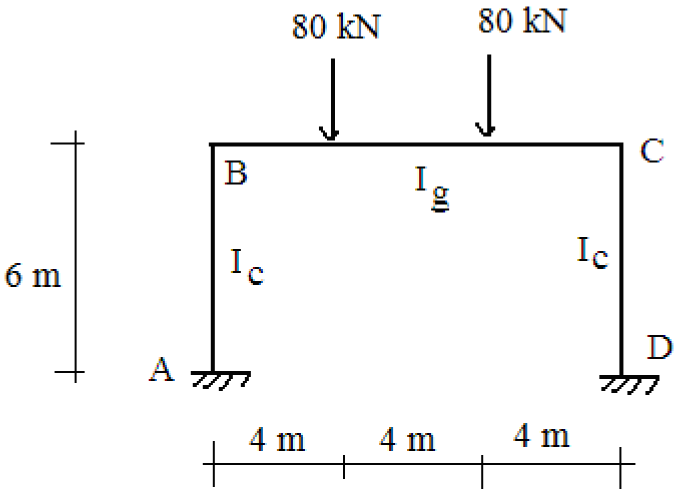
(b) Symmetrical loading- no sway

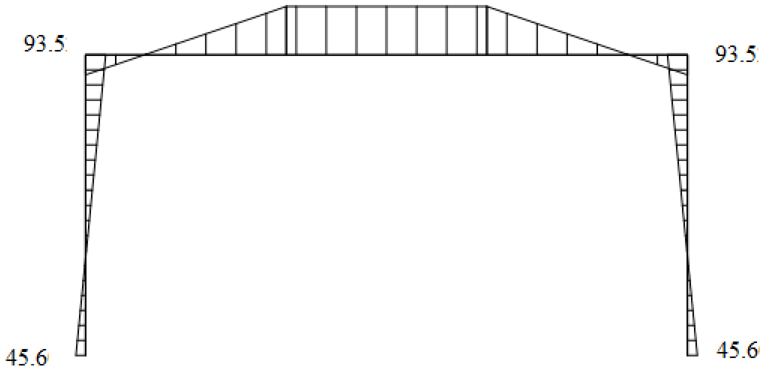




Moment diagram (kN-m)

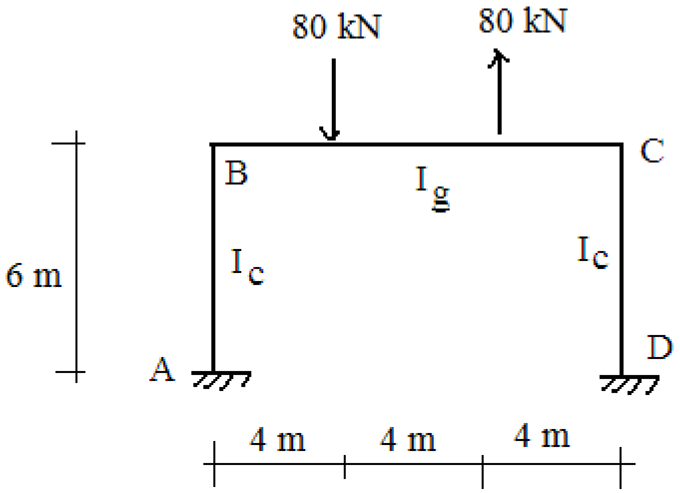
(c) Symmetrical loading- no sway

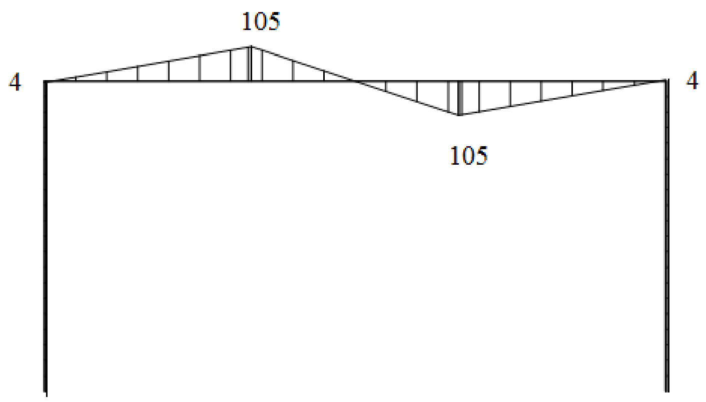




Moment diagram (kN-m

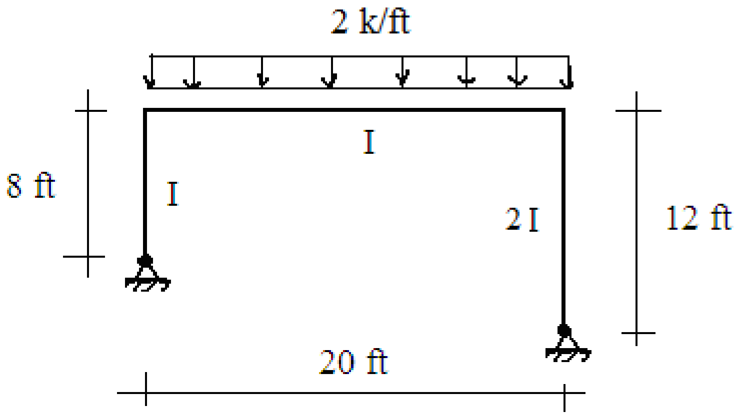
(d)

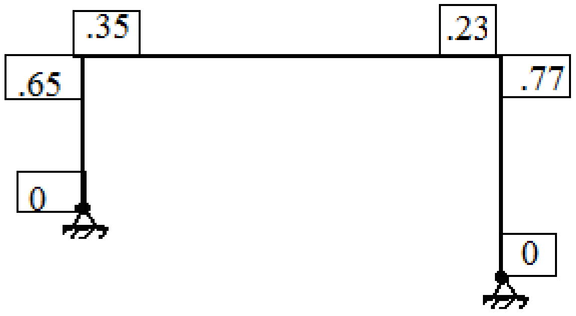


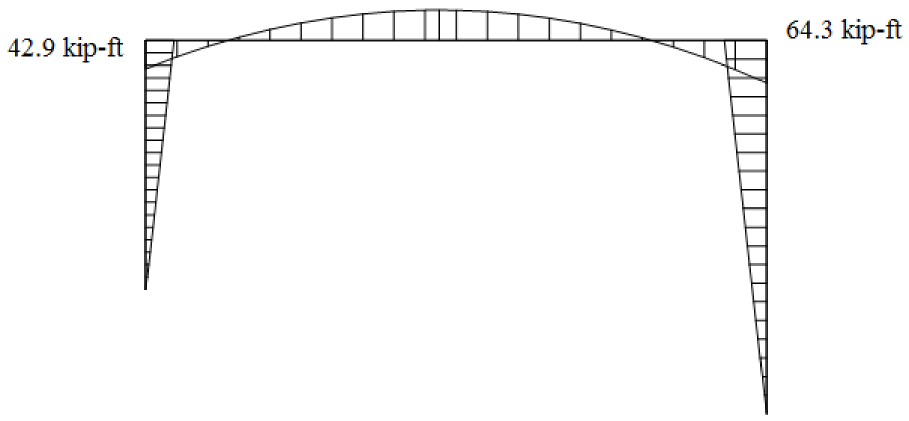


Moment diagram (kN-m)

Problem 10.31
Solve for the bending moments.



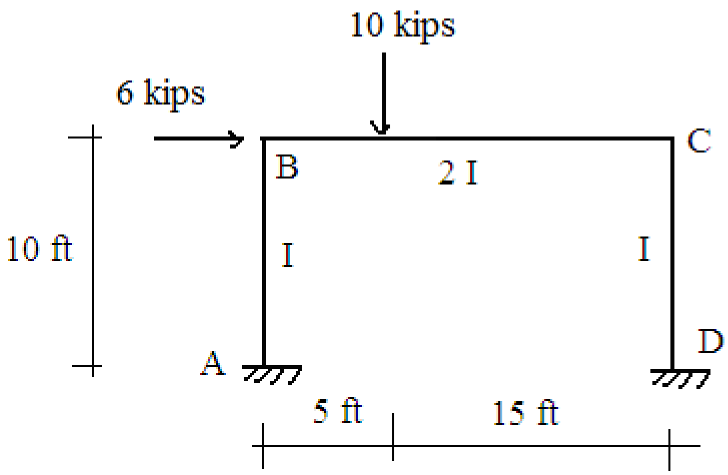


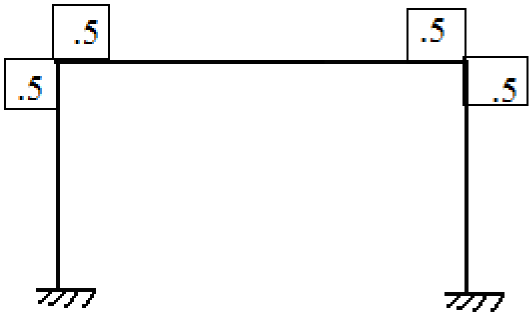


Moment diagram

Problem 10.32

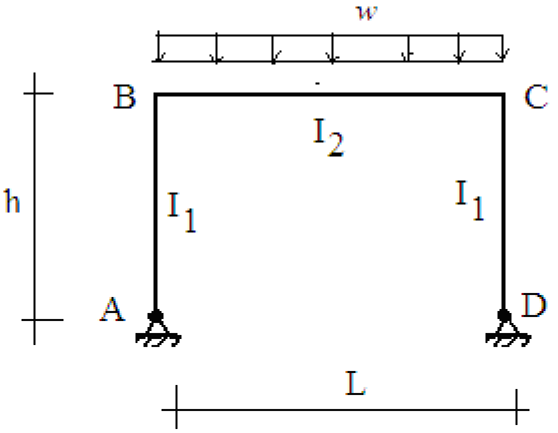
Determine the bending moment distribution.





Moment diagram (kip-ft)

Problem 10.33
Solve for the bending moments.



$I_1 = 200 \text{ in}^4$

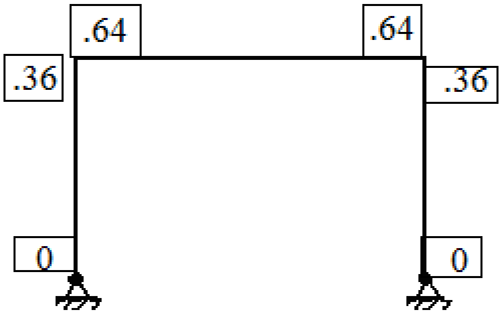
$I_2 = 400 \text{ in}^4$

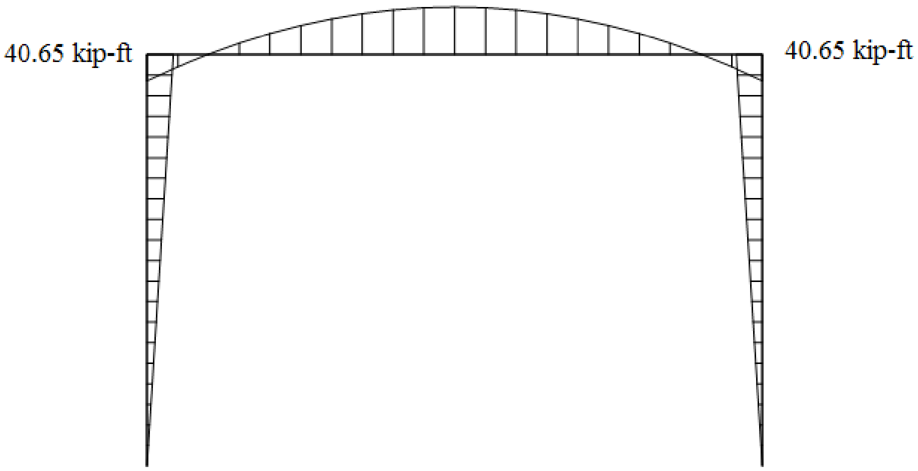
$L = 24 \text{ ft}$

$h = 16 \text{ ft}$

w=1.6 k/ft

Symmetrical loading- no sway

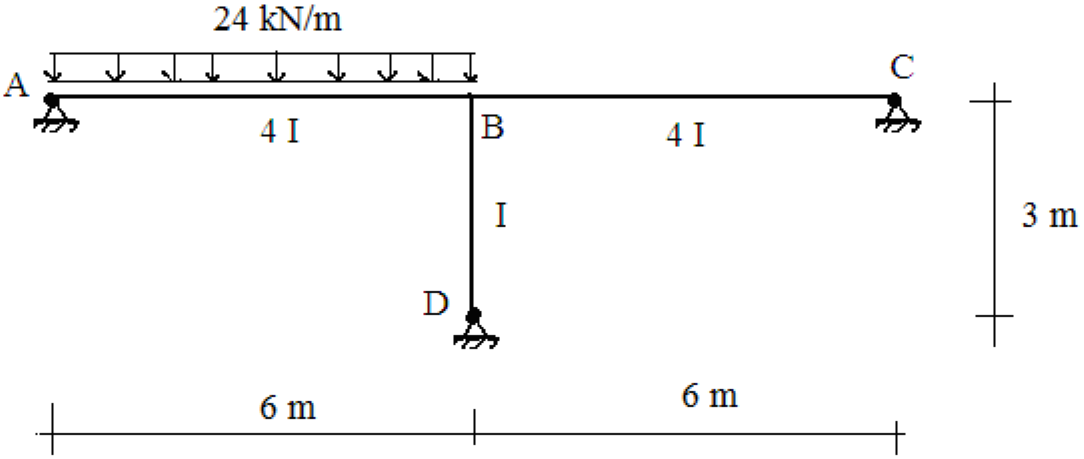




Moment diagram

Problem 10.34

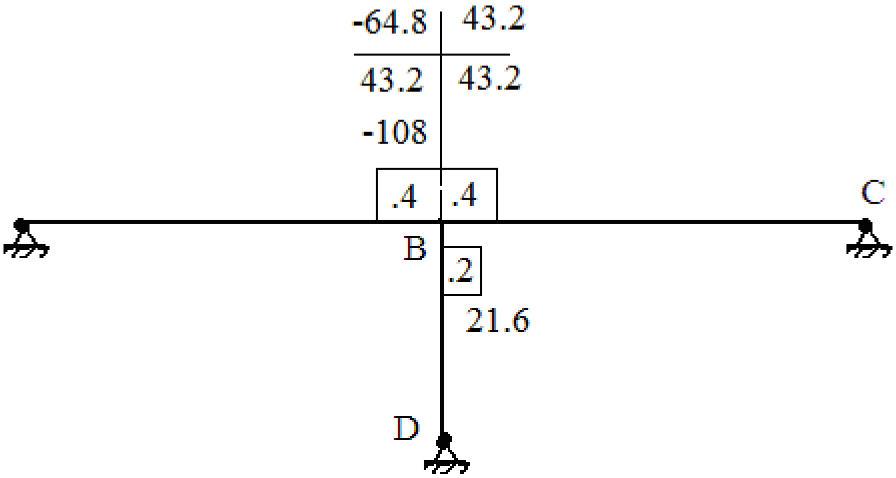
For the frame shown, determine the end moments and the reactions. Assume $E = 200\ \text{GPa}$, and $I = 40(10)^6\ \text{mm}^4$.

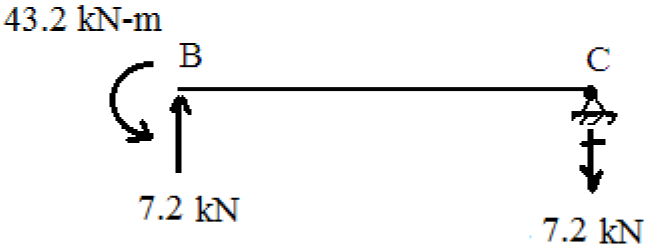


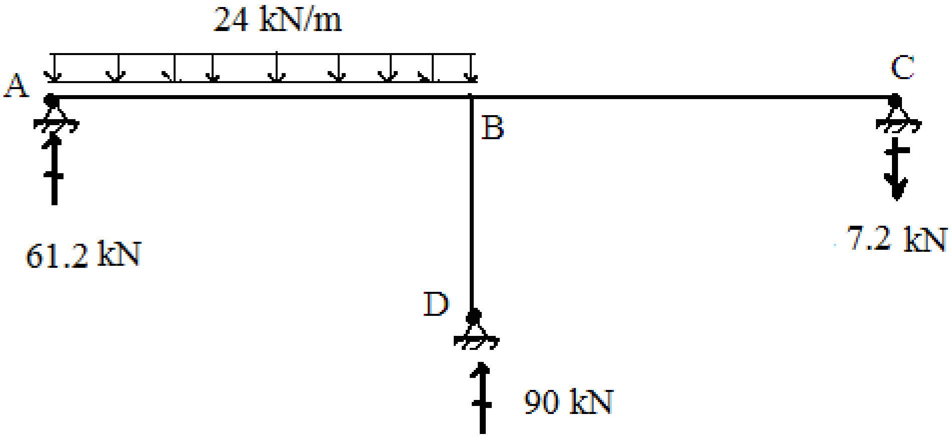
$$M^F{}_{AB} = 0$$

$$M^F{}_{BA} = -\frac{24(6)^2}{8} = -108\ \text{kN-m}$$

$$\text{Joint B}\begin{cases} \sum \frac{I}{L} = \frac{3}{4}(\frac{4I}{6}) + \frac{3}{4}(\frac{4I}{6}) + \frac{3}{4}(\frac{I}{3}) = \frac{5I}{4} \\ DF_{BA} = DF_{CB} = \frac{\frac{I}{2}}{\frac{5I}{4}} = 0.4 \qquad DF_{DB} = .2 \end{cases}$$

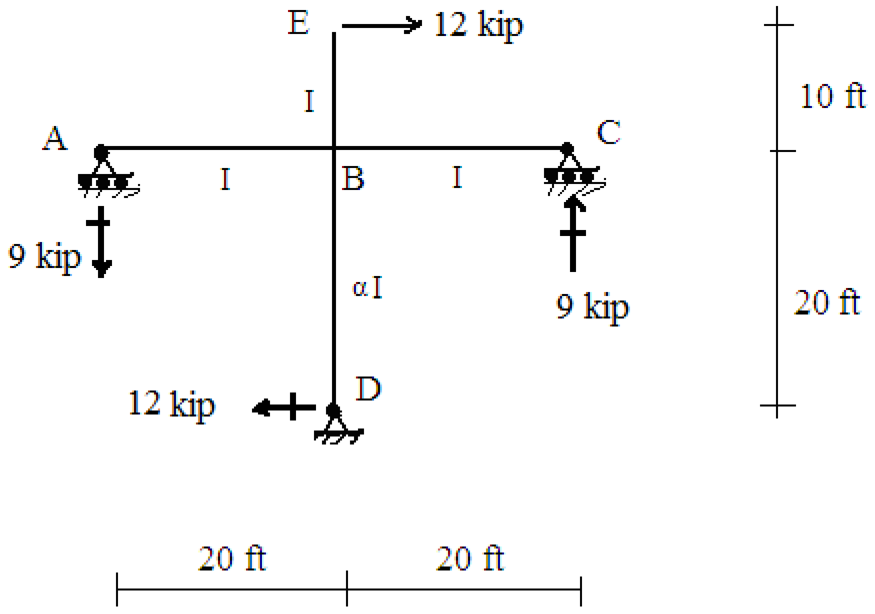




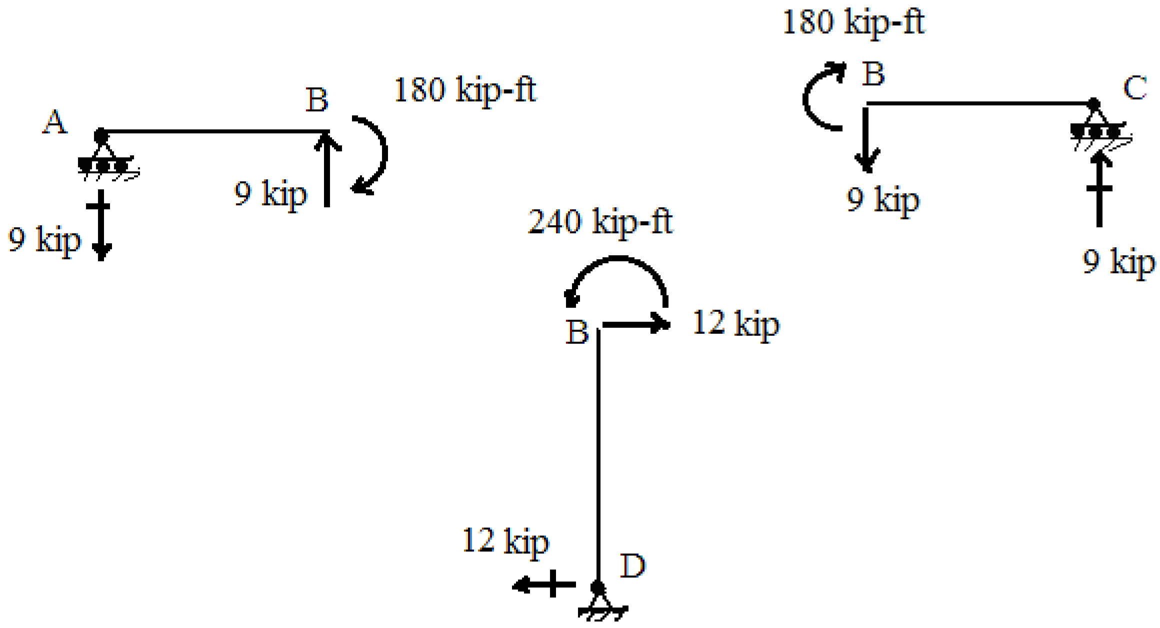


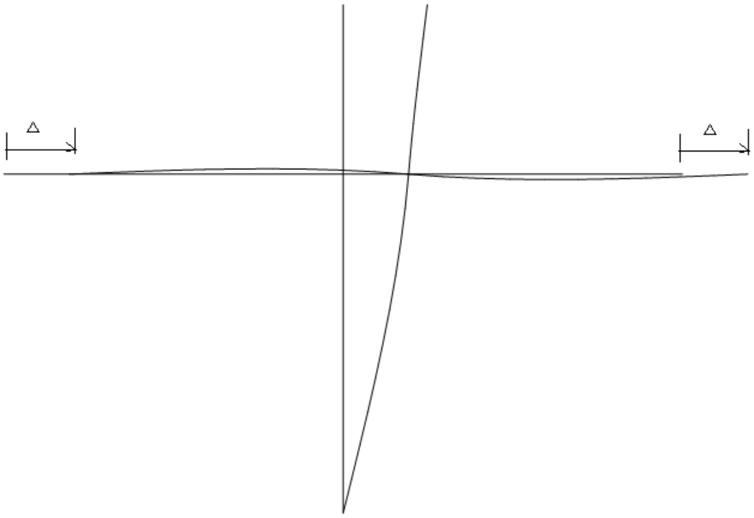
Problem 10.35

Determine analytic expression for the rotation and end moments at B. Take $I = 1000$ in^4, A=20 in^2 for all members, and α =1.0, 2.0, 5.0. Is there a upper limit for the end moment, M_{BD}?



M_{BD} is independent of α. $M_{BD} = 240$ kip-ft





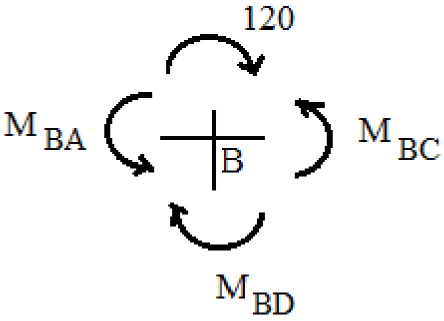
$$k_{AB} = k_{BC} = \frac{EI}{20} = k \qquad k_{BD} = \frac{\alpha I\ E}{20} = \alpha k \qquad \rho = \frac{\Delta}{20}$$

$$M_{BA\ \text{modified}} = 3(k)\ \theta_B$$

$$M_{BC\ \text{modified}} = 3(k)\ \theta_B$$

$$M_{BD\ \text{modified}} = 3(\alpha k)\ \theta_B + 3(\alpha k)\rho$$

$$M_{BA\ \text{modified}} = 3(k)\ \theta_B = -180 \qquad k\ \theta_B = -60$$



$$\sum_{\text{joint B}} M = 0$$

$$- 3(\alpha k)\ \theta_B + 3(\alpha k)\rho + 6(k)\ \theta_B - 120 = 0$$

$$M_{BD\ \text{modified}} = 3(\alpha)(\text{-}60) + 3(\alpha k)\rho = 240 \qquad k\ \rho = \frac{80 + 60\alpha}{\alpha}$$

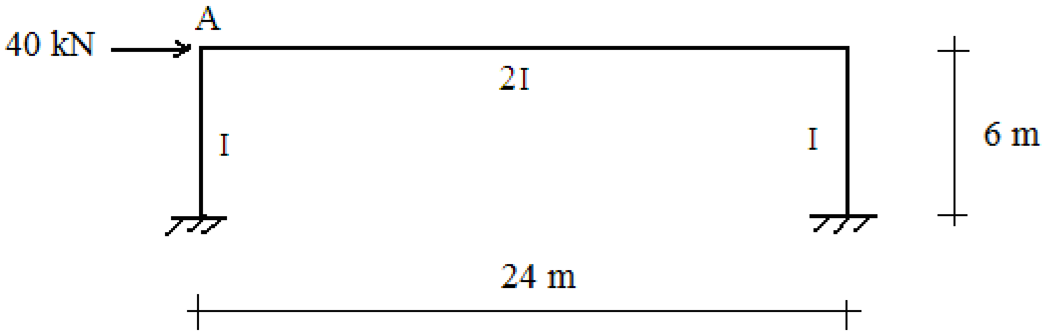
$$k = \frac{EI}{20} = \frac{29000(1000)}{20}\frac{1}{(12)^2} = 10069.4\ \text{k-ft}$$

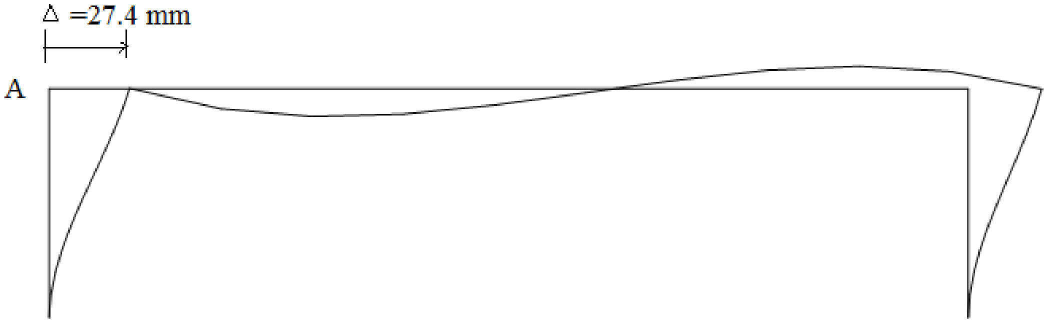
$$\Delta = \rho L = \frac{(80+60\alpha)(20)12}{\alpha\ k} = \begin{cases} 3.33 \text{ in} & \text{for } \alpha = 1.0 \\ 2.38 \text{ in} & \text{for } \alpha = 2.0 \\ 1.81 \text{ in} & \text{for } \alpha = 5.0 \end{cases}$$

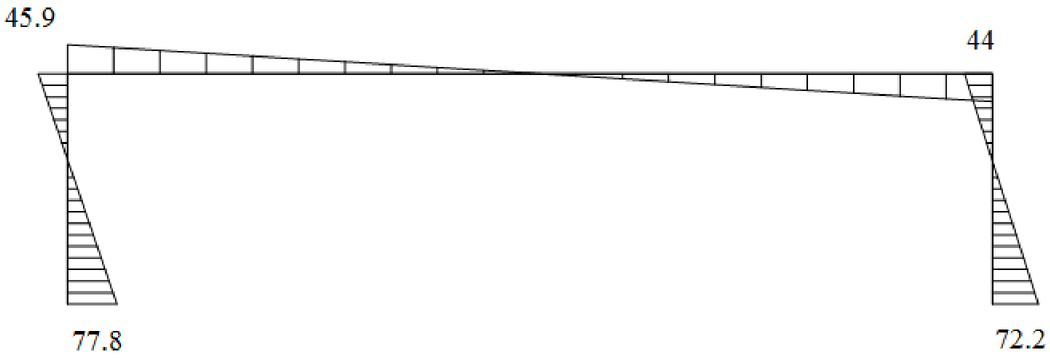
Problem 10.36

Compare the end moments and horizontal displacement at A for the rigid frames shown below. Check your results for parts (c) and (d) with a computer based analysis. Take $E = 200$ GPa , and $I = 120(10)^6\ mm^4$. $A = 10000\ mm^2$ for all members.

Case (a)

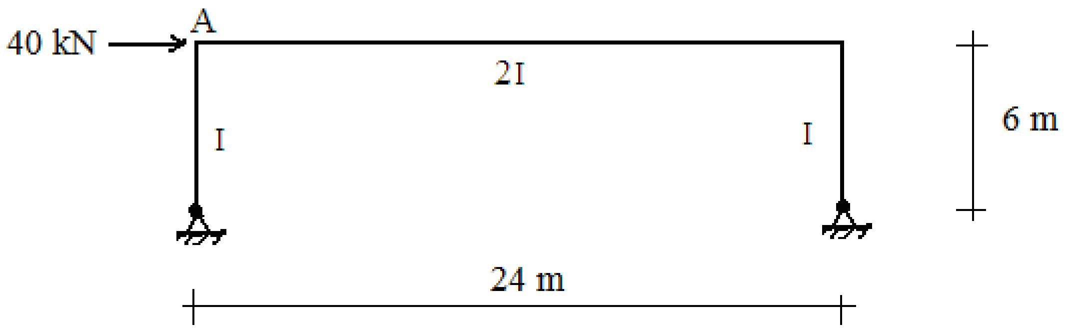


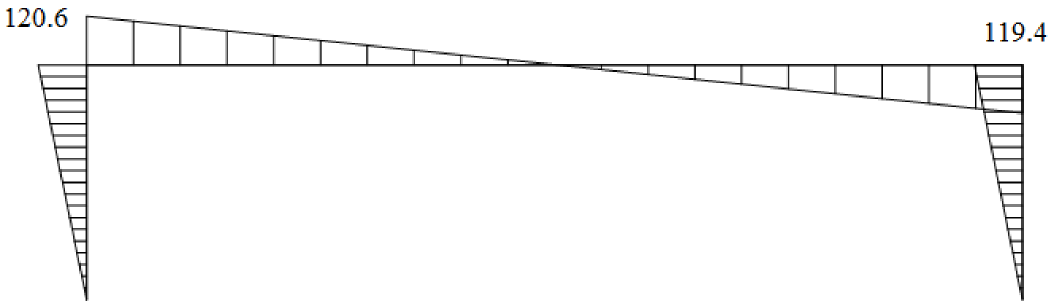




Moment diagram kN-m

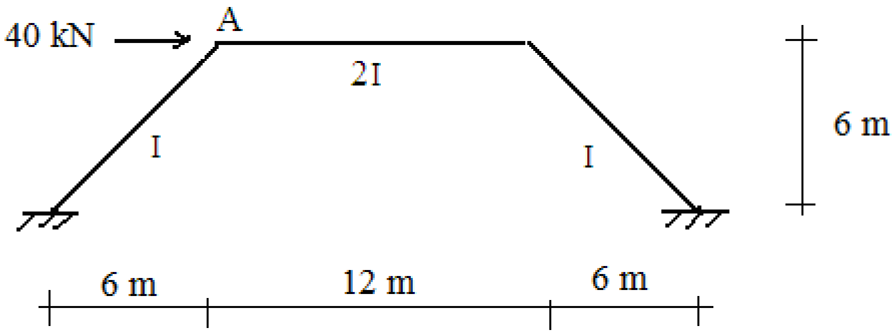
Case (b)

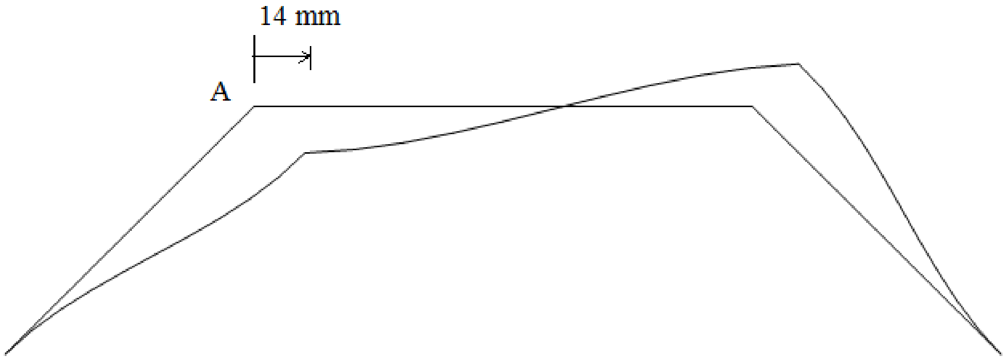


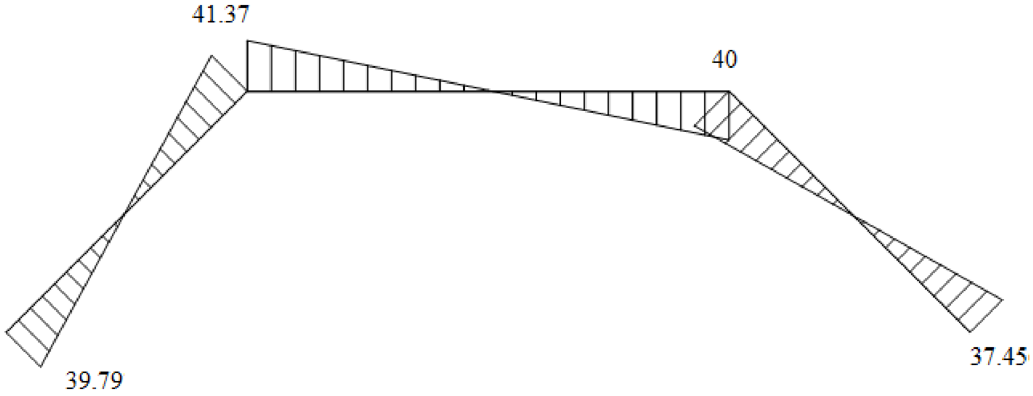


Moment diagram kN-m

Case (c)

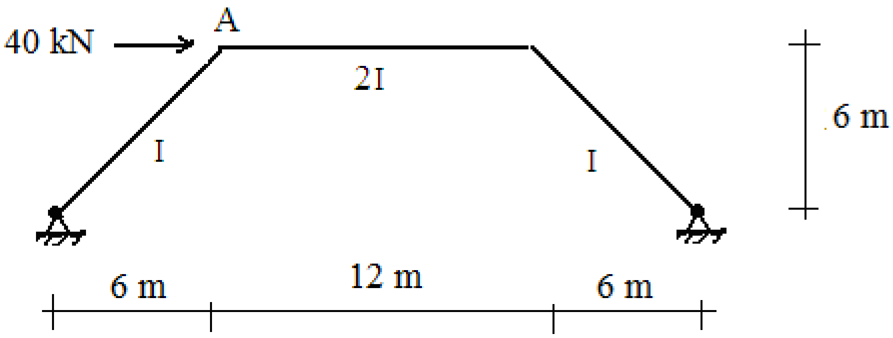


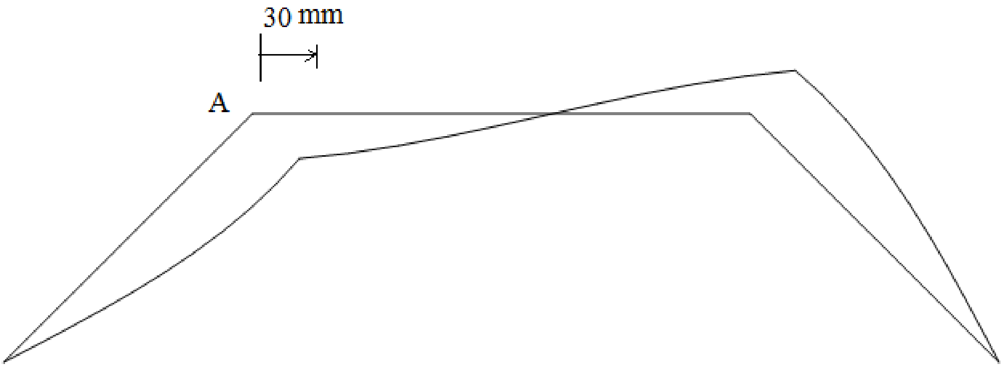


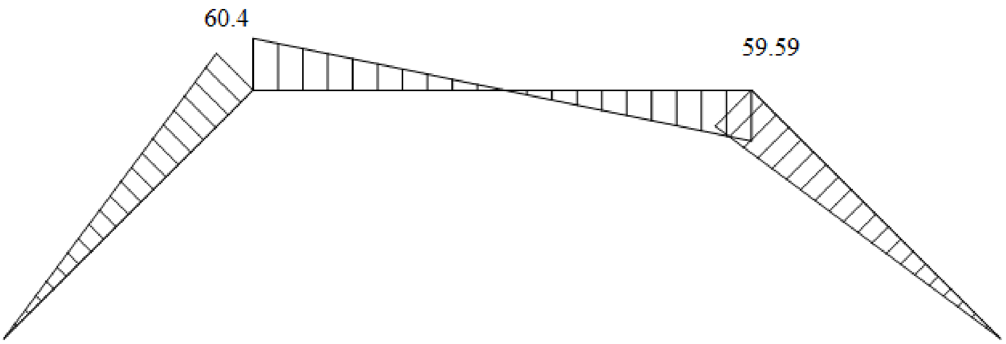


Moment diagram kN-m

Case (d)







Moment diagram kN-m